Complex Number Addition

MATH NSPIRED

TEACHER NOTES

Math Objectives

- Students will compute the sum of two complex numbers.
- Students will visualize and geometrically describe the sum of two complex numbers.
- Students will compute the absolute value and use trigonometry to find the argument of complex numbers.
- Students will compare the absolute values and arguments of two complex numbers to the absolute value and argument of their sum.
- Students will look for and express regularity in repeated reasoning (CCSS Mathematical Practice).
- Students will look for and make use of structure (CCSS Mathematical Practice).

Vocabulary

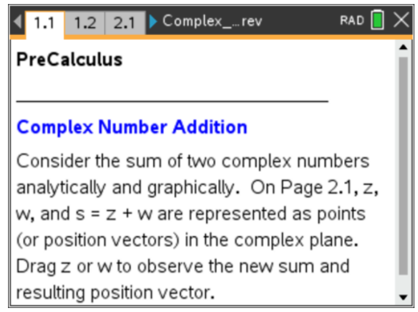
- complex number
- position vector
- absolute value or magnitude of a complex number
- argument of a complex number

About the Lesson

- This lesson involves the addition of two complex numbers.
- As a result, students will:
  - Compute the sum of specific complex numbers, make a general statement to describe this sum, and characterize the sum geometrically.
  - Compute the absolute value and argument of complex numbers.
  - Investigate and analyze the sum of two complex numbers with point representations that lie on the same line.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Live Presenter to demonstrate.
- Use Quick Poll to assess students' understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- Once a function has been graphed, the entry line can be shown by pressing ctrl G. The entry line can also be expanded or collapsed by clicking the chevron.

Lesson Files:

Student Activity

Complex_Number_Addition_Student.pdf

Complex_Number_Addition_Student.doc

TI-Nspire document

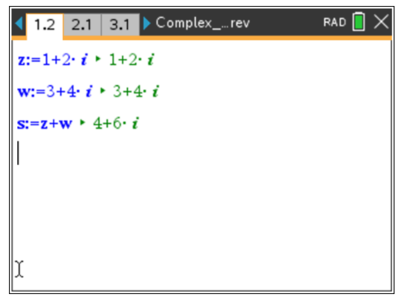
Complex_Number_Addition.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand ( ) getting ready to grab the point. Also, be sure that the word *point* appears, not the word *text*. Then press ctrl to grab the point and close the hand ( ).

Move to page 1.2.



1. This Notes page contains three interactive Math Boxes for the complex numbers $z,\ w,$ and the sum $s = z + w.$

   a. Redefine z and/or w as necessary to complete the following two tables. To redefine z or w, edit the Math Box following the assignment characters (i.e., $:=$).

Answer: Answers appear in the final row of each table.

z	$3+5i$	$-3-4i$	$11-11i$	$-5-6i$
w	$-4+7i$	$-2+6i$	$-11+12i$	$-7-9i$
$z+w$	$-1+12i$	$-5+2i$	i	$-12-15i$

z	$-\frac{1}{2}-\frac{3}{4}i$	$1-\sqrt{2}i$	$\frac{\sqrt{3}}{2}-3i$	$\frac{3}{5}-\frac{4}{5}i$
w	$1+\frac{1}{4}i$	$-1-\sqrt{2}i$	$\frac{\sqrt{3}}{2}+3i$	$\frac{2}{5}-\frac{4}{5}i$
$z+w$	$\frac{1}{2}-\frac{1}{2}i$	$-2\sqrt{2}i$	$\sqrt{3}$	$1-\frac{8}{5}i$

   b. Let $z = a + bi$ and $w = c + di.$ Explain in words how the complex numbers are added in terms of the real parts and the imaginary parts.

Sample Answers: The sum of two complex numbers is the sum of the real parts plus the sum of the imaginary parts.

c. Let $z = a + bi$ and $w = c + di$. Write the sum, $s = z + w$, symbolically in terms of the constants a, b, c, and d.

Answer: $s = z + w = (a + bi) + (c + di) = (a + b) + (c + d)i$

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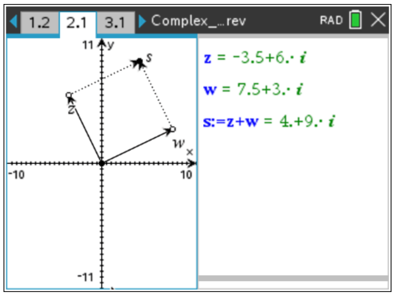


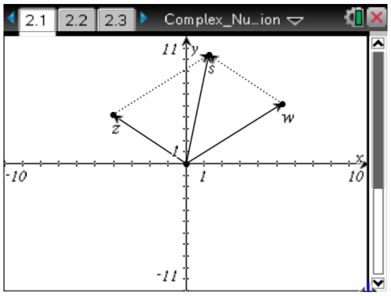
2. In the left panel, the complex numbers z and w are represented by points and position vectors in the plane. Point s represents the sum of these two complex numbers. Drag either point z or point w, and the sum is automatically computed and updated. The right panel displays the actual values for z, w, and s.

   a. Drag points z and w around the plane, and observe the results. Explain addition of complex numbers geometrically.

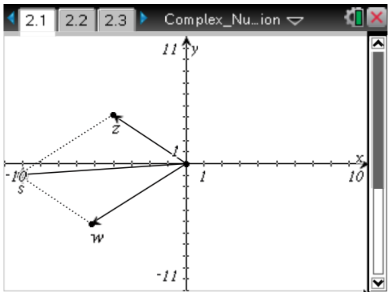
Answer: Addition of two complex numbers represented as points in the plane can be interpreted by constructing a parallelogram. Their sum is represented by the point at the end of the diagonal of the parallelogram which passes through the initial points (or intersection points) of the two vectors.

   b. Position point z in the second quadrant and point w in the first quadrant. On the first set of axes below, sketch a figure representing the resulting sum $s = z + w$. On the second set of axes below, sketch a figure that you think might represent the difference $d = z - w$. Drag and position point w to confirm your hypothesis. Hint: $d = z + (-w)$.

Sample Answers:

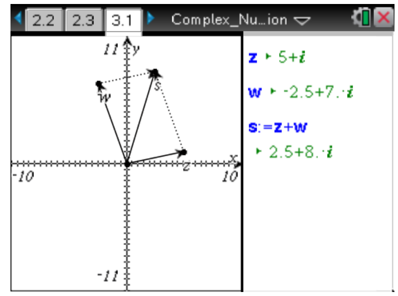


$s = z + w$



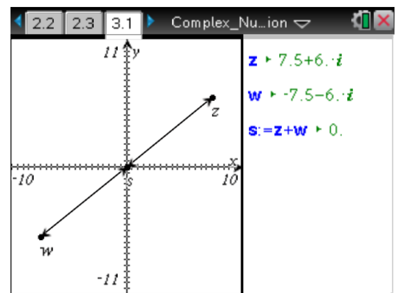
$d = z - w$

Move to page 3.1.

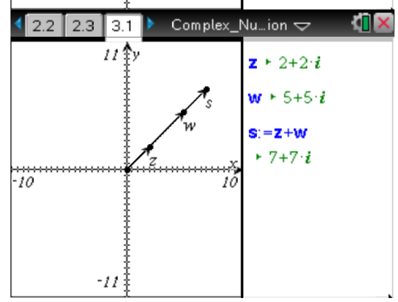


3. This page is a copy of Page 2.1 such that the real and imaginary parts of points z and w move only in increments of 0.25.

   a. Drag and position point z and/or point w so the sum is 0—that is, $s=0+0i$ and is represented by a point at the origin. Explain the relationship between points z and w.



<u>Sample Answers:</u> The points representing the complex numbers z and w lie on the same line through the origin, in opposite directions, and they appear to be the same distance from the origin. The complex numbers z and w are additive inverses, $z=-w$.



   b. Drag and position point z and point w such that $z=2+2i$ and $w=5+5i$. Find the sum s, and explain the relationship between the points representing z, w, and s.

<u>Answer:</u> $s=z+w=(2+2i)+(5+5i)=7+7i$

The points representing z, w, and s all lie on the same line through the origin.

   c. The absolute value or magnitude of a complex number $z=a+bi$ is $|z|=\sqrt{a^2+b^2}$. Find the absolute value of z, w, and s in part 3b, and explain how these three values are related.

<u>Answer:</u>

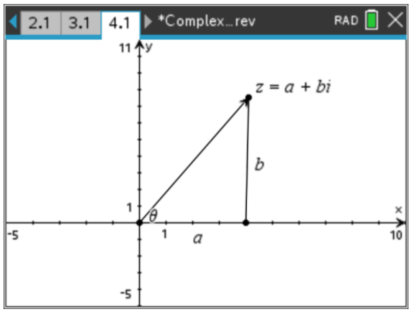
$$|z|=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$$
$$|w|=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}$$
$$|s|=\sqrt{7^2+7^2}=\sqrt{98}=7\sqrt{2}$$
$$|z|+|w|=2\sqrt{2}+5\sqrt{2}=7\sqrt{2}=|s|$$

Teacher Tip: Ask students whether this relationship is true for all complex numbers z and w.

TI-Nspire Navigator Opportunity: *Quick Poll and Screen Capture*
See Note 1 at the end of this lesson.

The argument of a complex number $z = a + bi$ is the angle, θ, (in radians) formed between the positive real axis and the position vector representing z. See the figure to the right. The angle is positive if measured counterclockwise from the positive real axis. Recall, $\tan\theta = \frac{b}{a}$.

d. Describe a method to find the argument of the complex number z in part 3b above. Find the actual argument for z, w, and s in part 3b. Explain how these three arguments are related.



<u>**Answer:**</u> $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ if the point representing z is in the first quadrant or if $a > 0$.

Argument for z: $\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

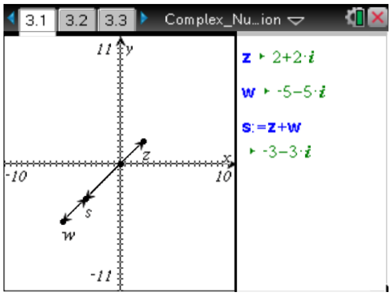
Argument for w: $\tan\theta = \frac{5}{5} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

Argument for s: $\tan\theta = \frac{7}{7} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

In this case, the argument of all three complex numbers is the same.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

4. Drag and position point z and point w such that $z = 2 + 2i$ and $w = -5 - 5i$.



   a. Find the sum s, and explain the relationship between the points representing z, w, and s.

<u>**Answer:**</u> $s = z + w = (2 + 2i) + (-5 - 5i) = -3 - 3i$
The points representing z, w, and s all lie on the same line through the origin.

b. Find the absolute value of z, w, and s in part 4a, and explain how these three values are related.

Answer:

$|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$|w| = \sqrt{(-5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$

$|s| = \sqrt{(-3)^3 + (-3)^3} = \sqrt{18} = 3\sqrt{2}$

In this example: $|s| = |w| - |z|$

c. Find the argument of points z and w. How are they related?

Answer: Argument for z: $\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

Argument for w: $\tan\theta = \frac{-5}{-5} = 1$ and θ is in the third quadrant. Therefore, $\theta = \frac{5\pi}{4}$.

$\arg(w) = \arg(z) + \pi$

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 3 at the end of this lesson.

Extensions

1. Ask students to position the points representing point z and point w such that the parallelogram is a square. Consider the absolute value and argument for point z and point w in this case.
2. Ask students to investigate and conjecture about the relationship between $|s|$, $|z|$, and $|w|$ when the points representing z and w do not fall on the same line through the origin.
3. Ask students to construct a piecewise-defined function for the argument of a complex number $z = a + bi$ that depends upon the signs and values of a and b.

Teacher Tip: The complex plane can be thought of as a modified Cartesian coordinate system and consists of a horizontal, or real, axis and a vertical, or imaginary, axis. The complex number $z = a + bi$ is represented in the plane by the point (a, b).

Teacher Tip: The absolute value of a real number and a complex number have the same geometric meaning – the distance from the origin

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to:

- Compute and visualize the sum of two complex numbers.
- Understand the concepts of the absolute value and argument of a complex number.

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Note 1

Question 3c, Name of Feature: Quick Poll and Screen Capture

Ask students if this relationship is always true. Use Screen Capture to consider possible counter-examples.

Note 2

Question 3d, Name of Feature: Quick Poll

Ask students for the arguments of z, w, and s.

Note 3

Question 4c, Name of Feature: Quick Poll

Ask students for the arguments of z and w.