

# Approximate Areas

## Teacher Notes & Answers

7 8 9 10 11 12



TI-30XPlus  
MathPrint™



Worksheet



Student



45 min

## Choosing a shape

### Teacher Notes:

Students are provided with a range of strategies to determine the area under a curve. Starting with a simple triangle, left and right bound rectangles, an averaging of the left and right bound rectangles (trapeziums) students see how better approximations for the area under a curve are formed using different shapes. Students then explore how increasing the number of divisions (rectangles and trapeziums) produces a limiting value for the area. Finally, students are provided with an 'unfamiliar' option: midpoints. The purpose of introducing midpoints is for students to consider the shape of the curve and to evaluate students understanding of the previous calculations and calculator functionality.

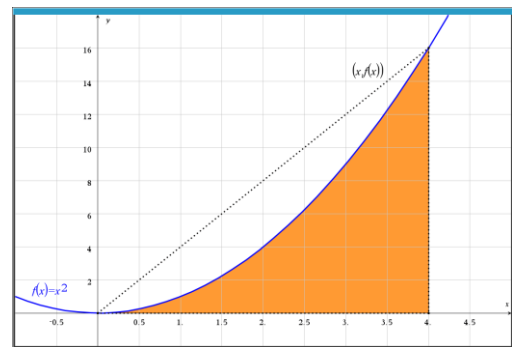
A set of Power Point slides are also provided for presentation purposes. The extension section is option and can easily be cut out. The function tool, lists and the sum command are used on the calculator enabling students to work efficiently.

The area bounded by the curve  $f(x) = x^2$ , the  $x$  axis and the line  $x = 4$  is shown opposite.

### Question: 1.

The area can be approximated by a single triangle with vertices:  $(0, 0)$ ,  $(0, 4)$  and  $(4, 16)$ . Determine the area of the triangle.

**Answer:**  $\frac{1}{2} \times 4 \times 16 = 32$



### Question: 2.

Another way to estimate the area is to use a series of equally spaced rectangles. (Shown opposite)

i) What is the width of each rectangle?

**Answer:** 1 unit

ii) The height of each rectangle is defined by the point where the top right corner intersects the curve. Determine the height of each rectangle and hence the approximate area.

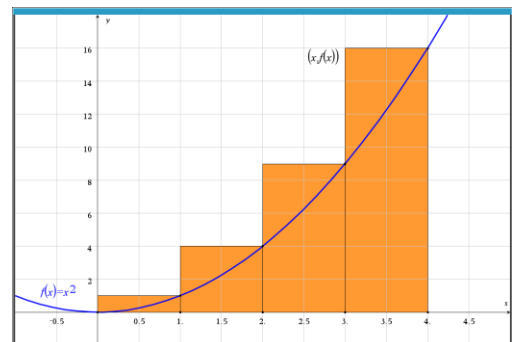
**Answer:**  $1 + 4 + 9 + 16 = 30$ . or  $f(1) + f(2) + f(3) + f(4) = 30$

iii) Which approximation, triangle or rectangle, is closest to the actual area? (Explain).

**Answer:** Both estimates are greater than the actual area, therefore the actual area must be less than 30, so it follows that the right bound rectangles provide the better estimate.

iv) Would your answer to the previous question (Part iii) be the same if the area was being estimated between  $x = 0$  and  $x = 3$  using three right bound rectangles versus a single triangle?

**Answer:** Yes and No. Yes: Both estimates are still greater than the actual area. No: The triangle estimate would be:  $\frac{1}{2} \times 3 \times 9 = 13.5$  and right bound rectangles would be:  $f(1) + f(2) + f(3) = 14$ . The actual area must be less than 13.5. In this case, the triangle provides a better estimate of the actual area.



**Question: 3.**

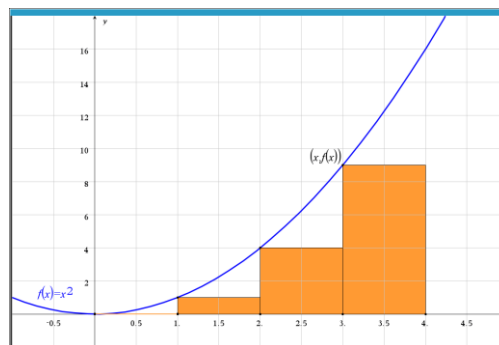
The rectangles shown opposite are referred to as “left bound”. The height of each rectangle is defined by the upper-left corner.

- i) How will this area estimate compare with previous estimates and the actual area?

**Answer:** This estimate will be smaller than all previous estimates and smaller than the actual area.

- ii) Determine the height of each rectangle and hence the approximate area according to the left bound rectangles.

**Answer:**  $0 + 1 + 4 + 9 = 14$  or  $f(0) + f(1) + f(2) + f(3) = 14$

**Question: 4.**

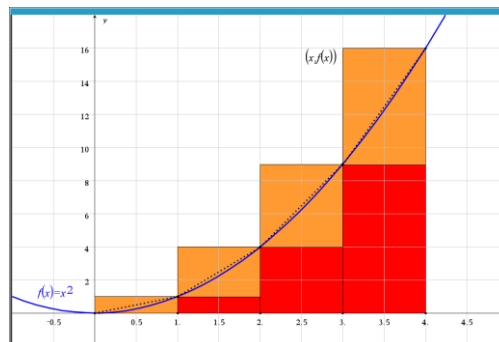
The graph opposite shows both the left and right bound rectangles. Averaging the two areas will provide a better approximation. Dotted line segments have been included to illustrate how this averaging may be visualised.

- i) What shapes are formed geometrically by the averaging of the left and right rectangles?

**Answer:** Generally, the shapes will be trapeziums. In this example the very first shape is a triangle since the function passes through the origin.

- ii) Determine the average of the left and right bound rectangles and comment on the result.

**Answer:** Average:  $(14 + 30) \div 2 = 22$ . The calculated area will be much closer to the actual area as the over and under estimates partially ‘cancel’ one another out. The estimate will be slightly more than the actual area (as the curve is concave up).



The calculations required for three regions can be done quite easily and quickly by hand, if more divisions are created the calculations become repetitive and time consuming! The instructions below demonstrate how to perform these types of calculations efficiently on your calculator. You can also scan the QR code to watch a video tutorial!

**Using your calculator – Lists**

The first step is to enter the function. Press:

expr-eval

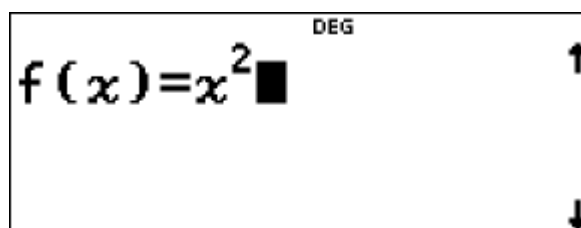
**table**

... select **Add/Edit Function**

Enter the function  $f(x) = x^2$  and then press:

**enter**

... now quit from the function menu.



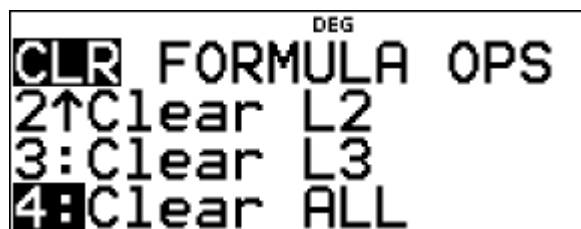
The calculator's lists can be used to perform multiple calculations quickly and efficiently. Make sure the lists are cleared and ready for the calculations.

stat-reg/distr stat-reg/distr

**data**

**data**

... clear lists as necessary.



The next step is to generate a list of the x values that form the boundary conditions. For small lists, the entries can be done manually, for larger lists, use the sequence command.

stat-reg/distr

**data** navigate to **OPS** and select **Sequence**

The sequence of numbers will go in List 1 (L1)

The sequence formula is simply:  $x$

To produce left bound rectangles of width 0.5, the x-values need to be: 0, 0.5, 1, 1.5, 2, 2.5, 3 and 3.5

Enter the settings as shown opposite to produce this result, then press:

**enter**

... to generate the sequence (list).

```

DEG
SEQUENCE FILL
FILL LIST: [ ] L2 L3
1 ≤ dim(list) ≤ 50
  
```

Navigate across to List 2 (L2). Then press:

stat-reg/distr

**data** ... select **Formula** then **Add/Edit Formula**

The **Table** key can be used to access the functions and the **Data** key to access list names.

Match the formula shown opposite for the values in List 2.

```

DEG
EXPR IN X: X
START X: 0
END X: 3.5
STEP SIZE: 0.5
SEQUENCE FILL
  
```

To sum all the values in List 2, press:

stat-reg/distr

**data** ... select **Ops** then **Sum List**.

```

L1 L2 DEG L3
0
0.5
1
1.5
L2=f(L1)
  
```

Make sure List 2 is selected for the sum.

The sum is displayed on screen (35 in this example) and can be stored in one of the calculator memories.

Store the result in 'a'.

```

DEG
CLR FORMULA OPS
2↑Sort L9-Sm...
3: Sequence...
4 Sum List...
  
```

```

DEG
SUM LIST
SUM OF LIST=35
STORE: No X Y Z T [ ] b c d
DONE
  
```

Recall that the columns in this example are only 0.5 units wide, so the area is equal to:  $a \div 2$ .

```

DEG
a/2
  
```

### Question: 5.

Use your calculator to determine the approximate area for the same function and region from the previous question, using a column width of 0.5 units for:

i) Left Bound rectangles

**Answer:** [Starts with  $x = 0$ ] Area  $\approx 17.5$ .

ii) Right Bound rectangles

**Answer:** [Starts with  $x = 0$ ] Area  $\approx 25.5$ .

iii) Trapeziums

**Answer:** [Average of previous two results.] Area  $\approx 21.5$ .

**Question: 6.**

Use your calculator to determine the approximate area for the same function and region from the previous question, using a column width of 0.1 units for:

- i) Left Bound rectangles  
**Answer:** [Starts with  $x = 0$ ] Area  $\approx 20.54$ .
- ii) Right Bound rectangles  
**Answer:** [Starts with  $x = 0$ ] Area  $\approx 22.14$ .
- iv) Trapeziums  
**Answer:** [Average of previous two results.] Area  $\approx 21.34$ .

**Question: 7.**

Use your calculator to determine the approximate area bounded by the curve:  $y = 4 - x^2$  and the  $x$  axis using a column width of 0.1 units for:

- i) Left Bound rectangles  
**Answer:** [Starts with  $x = -2$  and ends with 1.9] Area  $\approx 10.66$ .
- ii) Right Bound rectangles  
**Answer:** [Starts with  $x = -1.9$  and ends with 2.0] Area  $\approx 10.66$ .
- iii) Explain why the left and right bound rectangle calculations are the same.  
**Answer:** The first column height for LBR is 0, similarly with the last column for RBR, all other columns are the same.
- iv) Determine the approximate area using trapeziums of width 0.1 units.  
**Answer:** The answer will be the same since the trapeziums are simply an average of the left and right bound rectangles.

**Question: 8.**

Use your calculator to determine the approximate area bounded by the curve:  $y = 4 - x^2$ , the  $y$  and  $x$  axes where  $x \geq 0$ , using a column width of 0.1 units for:

- i) Left Bound rectangles  
**Answer:** [Starts with  $x = 0$  and ends with 1.9] Area  $\approx 5.53$ .
- ii) Right Bound rectangles  
**Answer:** [Starts with  $x = 0.1$  and ends with 2.0] Area  $\approx 5.13$ .
- iii) Explain why the left bound area approximation is larger than the right bound.  
**Answer:** The curve is concave down (convex) so the left bound rectangles are larger than the right bound.  
**Teacher Notes:** Students may choose to use a diagram to illustrate the difference, the important factor addressed in this question is that students understand how concavity impacts the area approximations.

**Extension – Alternative Approximations**

Another approach to approximating the area under a curve is to use the **midpoint rule**. The left and right bound rectangles use the bordering values according to the abscissa ( $x$  coordinate), the midpoint rule determines the height of each rectangle based on the midpoint of the abscissa values. For example, finding the approximate area under the curve  $y = x^2$  using four intervals, rectangle heights were determined using either left bound:  $\{f(0), f(1), f(2), f(3)\}$  or right bound:  $\{f(1), f(2), f(3), f(4)\}$ . For the midpoint rule, heights are determined using:  $\{f(0.5), f(1.5), f(2.5), f(3.5)\}$ .

**Question: 9.**

For the following question, the area being explored is bounded by the curve:  $y = x^2$ , the  $x$  axis and the line  $x = 4$ .

- i) Use the midpoint rule to determine the approximate area using a column width of 1 unit.

**Answer:** [Starts with  $x = 0.5$  and ends with 3.5] Area  $\approx 23.0$ .

- ii) Use the midpoint rule to determine the approximate area using a column width of 0.1 units.

**Answer:** [Starts with  $x = 0.05$  and ends with 3.95] Area  $\approx 21.33$ .

- iii) Explain how the midpoint rule is different to the trapezoidal rule.

**Answer:** For the height of the function:

- Trapezoidal rule uses:  $\frac{f(a) + f(b)}{2}$  (average value of the function)
- Midpoint rule uses:  $f\left(\frac{a+b}{2}\right)$  (average value of the abscissa values)

Another approach to approximating areas is to use **Simpson's rule**. This rule is quite different:

$$S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n))$$

To perform this calculation 'automatically' using the lists requires a somewhat complicated formula. The first and last terms:  $f(x_0)$  and  $f(x_n)$  are left out of the automated calculations. This means the only thing that needs to be sorted is multiplying by 4 for odd terms and 2 for even terms. To achieve this List 1 will be used to store the term number, List 2 for the abscissa and List 3 for the evaluation of  $f(x)$ .

List 1: Sequence: 1, 2, ... 39

List 2: Sequence: 0.1, 0.2, ... 3.9

List 3: Formula:  $4 \times \text{mod}(L1, 2) \times f(L2) + 2 \times \text{mod}(L1 - 1, 2) \times f(L2)$

The modular arithmetic command:  $\text{mod}(\#, \#)$  is available from the Math > Number menu.  $\text{Mod}(a, b)$  produces the remainder when  $a$  is divided by  $b$ , so  $\text{mod}(1, 2)$ ,  $\text{mod}(3, 2)$ ,  $\text{mod}(5, 2)$  ... each produce a result of 1.

In contrast,  $\text{mod}(2, 2)$ ,  $\text{mod}(4, 2)$ ,  $\text{mod}(6, 2)$  ... each produce a result of 0. The overall result of the formula is that when List 1 is odd, the result will be  $4 \times f(L2)$  and when List 1 is even the result will be  $2 \times f(L2)$ .

Once the sum of List 3 has been calculated,  $f(x_0)$  and  $f(x_n)$  can be added to the result and then multiplied by the appropriate fraction to compute the area estimate.

**Question: 10.**

For the following question, the area being explored is bounded by the curve:  $y = x^2$ , the  $x$  axis and the line  $x = 4$ .

- i) Use the table to help compute the approximate area using Simpson's rule with a column width of 0.5 units.

$n$	0	1	2	3	4	5	6	7	8
$x$	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x_n)$	0	0.25	1	1.25	4	6.25	9	12.25	16

**Answer:**

$$\begin{aligned}
 S_n &= \frac{0.5}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)) \\
 &= \frac{1}{6} (0 + 4(0.25) + 2(1) + 4(2.25) + 2(4) + 4(6.25) + 2(9) + 4(12.25) + 16) \\
 &= \frac{1}{6} (0 + 1 + 2 + 9 + 8 + 25 + 18 + 49 + 16) \\
 &= 21.3
 \end{aligned}$$

ii) Use Simpsons rule to determine the approximate area using a column width of 0.1 units.

**Answer:** 21.3\* [Note: This is exactly the same answer as part(i) and the actual integral!]

List 1 contains the term number.

List 2 contains the abscissa value

List 3 contains the formula where  $f(x)=x^2$

L1	L2	DEG	L3
1	0.1		0.04
2	0.2		0.08
3	0.3		0.36
4	0.4		0.32

L3 = L2 \* mod(L1,2) \* f(L2) + 2

List 3 is summed and stored in 'a'.

DEG
SUM LIST
SUM OF LIST=624
STORE: No x y z t a b c d
DONE

The final sum is computed as per the formula.

DEG
$\frac{0.1}{3} * (f(0) + a + f(4))$
21.33333333

**Teacher Notes:** Students should see that whilst Simpson's rule is the most complicated, the approximation for this curve is 'remarkably' accurate! Other functions should also be explore, such as higher degree polynomials and trigonometric functions. Now that the calculations are set up, it is essentially only the function definition and List 2 that need changing.

Example:

The equation for the normal distribution with mean = 0 and standard deviation = 1 can be entered in f(x). The integral from 0 to 4 (used in previous question) remained the same, with 40 intervals.

DEG
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

The approximate area under the standard normal curve for  $x > 0$  is approximately  $\frac{1}{2}$ . The result compares very favourably with the computed probability: 0.4999683134406  
CAVEAT: These calculations do take a while for the calculator to process.

DEG
$\frac{0.1}{3} * (f(0) + a + f(4))$
0.499968325