String Graphs – Part 1

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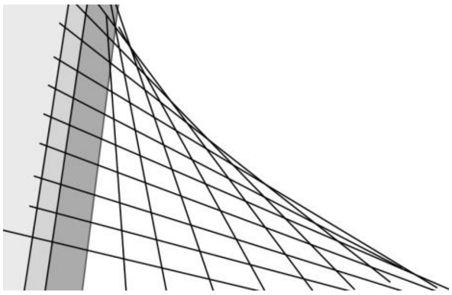
Answers

7 8 9 10 **11** 12

TI-Nspire | Investigation | Student | 45 min

Aims

- Determine a series of equations of straight lines to form a pattern similar to that formed by the cables on the Jerusalem Chords Bridge.
- Determine the parametric equation for the curve created by the successive intersection points.

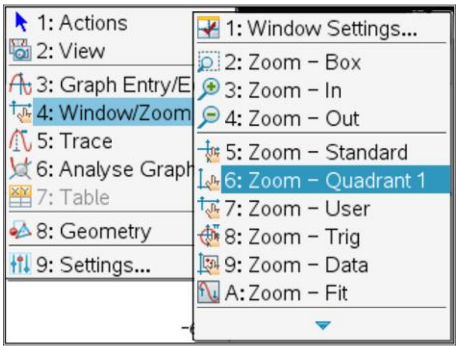


Determining Equations

Start a **new document** and insert a **Graph application**.

Use the **[Menu]** to adjust the window settings:

Window/Zoom > **Quadrant 1**.

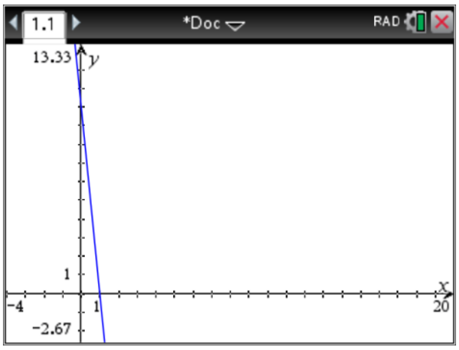


A series of straight line graphs will be constructed to form a string pattern similar to that on the Chords bridge.

The first straight line graph passes through the points:

(0, 10) & (1, 0)

The result is shown opposite. Use the questions to help determine the equation for this line and all subsequent lines.



The equation to any straight line can be expressed in the form: $y = mx + c$

m = gradient = $\frac{rise}{run}$

c = y-axis intercept

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Question: 1.

Determine the equation of this first line, passing through the points: (0, 10) & (1, 0)

a) Write down the y-axis intercept of the first line. (0, 10)

b) Calculate the gradient of the first line. $m = -10$

c) Write down the equation of the first line and graph it on the calculator. $y = -10x + 10$

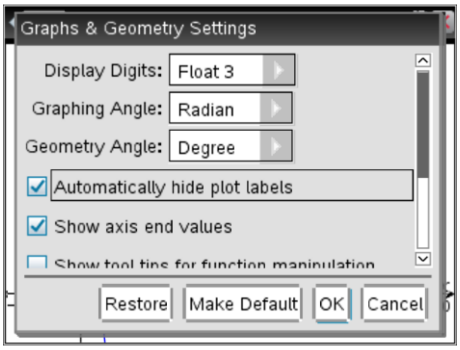
Once the first line is completed, try the second line.

The second straight line graph passes through the points:

(0, 9) & (2, 0)

As more graphs are added it may be desirable to remove the equation labels.

Settings > **Automatically hide plot labels**



Question: 2.

Determine the equation of the line, passing through the points: (0, 9) & (2, 0). $y = \frac{-9}{2}x + 9$

Question: 3.

Determine the gradient and y – intercept for the remaining straight lines in this family of lines. Record your results using exact values. Graph all 10 equations on the same set of axis.

Eqn. No.	Point 1	Point 2	Gradient	y-Intercept	Equation
1	(0, 10)	(1, 0)	-10	-10	$y = -10x + 10$
2	(0, 9)	(2, 0)	$-\frac{9}{2}$	-9	$y = -\frac{9}{2}x + 9$
3	(0, 8)	(3, 0)	$-\frac{8}{3}$	-8	$y = -\frac{8}{3}x + 8$
4	(0, 7)	(4, 0)	$-\frac{7}{4}$	-7	$y = -\frac{7}{4}x + 7$
5	(0, 6)	(5, 0)	$-\frac{6}{5}$	-6	$y = -\frac{6}{5}x + 6$
6	(0, 5)	(6, 0)	$-\frac{5}{6}$	-5	$y = -\frac{5}{6}x + 5$
7	(0, 4)	(7, 0)	$-\frac{4}{7}$	-4	$y = -\frac{4}{7}x + 4$

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8	(0, 3)	(8, 0)	$-\frac{3}{8}$	-3	$y=-\frac{3}{8}x+3$
9	(0, 2)	(9, 0)	$-\frac{2}{9}$	-2	$y=-\frac{2}{9}x+2$
10	(0, 1)	(10, 0)	$-\frac{1}{10}$	-1	$y=-\frac{1}{10}x+1$

A single equation can be determined to graph all 10 equations by using a parameter (t) for the equation number. Study each of your equations above and compare with the equation 'number'.

Question: 4.

The general equation is of the form: $y=\frac{a}{b}x+c$ where a, b and c are expressions in terms of t.

a) Determine an expression for a in terms of t. $a=t-11$ (Refer also to part d)

b) Determine an expression for b in terms of t. $b=t$ (Refer also to part d)

c) Determine an expression for c in terms of t. $c=11-t$

d) Write down the general equation for the family of straight lines. Verify your equation by substituting a range of values for t and comparing with the corresponding original equation.

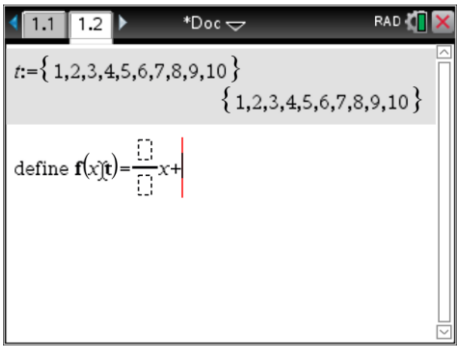
$y=\frac{t-11}{t}x+11-t$ Students may also have $y=\frac{11-t}{-t}x+11-t$ however they should be careful answering as: $y=-\frac{11-t}{t}x+11-t$ as the question required the form: $y=\frac{a}{b}x+c$

Insert a Calculator application and define t as the set of integers: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Define your general equation in terms of the variable x and parameter t.

Return to the Graph application and graph your function:

$f(x,t)$



Question: 5.

Describe the shape of the curve formed by the family of straight lines.

Initially the curve may appear like a hyperbola; however the curve is a parabolic.
There are several ways this can be illustrated. Part 2 of this investigation (separate activity) shows that this curve is a parabola rotated 45°.

Graph the following extended family of straight lines: $f(x,t)$, $f(x,t+10)$ and $f(x,t-10)$.
To see the full effect, zoom out using the zoom out tool in the Window / Zoom menu and place the magnifying glass close to the centre of the screen.

Question: 6.

Describe the shape of the curve formed by the extended family of straight lines.
The curve is parabolic. (As per Question 5)

Finding a Locus

The points of intersection between successive equations can be used to produce the curve where infinitely many straight lines are generated.[1]

Question: 7.

Show that the first two lines passing through (0, 10) & (1, 0) and (0, 9) & (2, 0) intersect when:

$x=\frac{2}{11}$ and $y=\frac{90}{11}$.

Eqn 1: $y=-10x+10$

Eqn 2: $y=-\frac{9}{2}x+9$

$$-10x+10=-\frac{9}{2}x+9$$
$$10-9=\left(-\frac{9}{2}+10\right)x$$
$$x=\frac{2}{11}$$

By substitution:
$$y=-10\left(\frac{2}{11}\right)+10$$
$$y=\frac{90}{11}$$

Question: 8.

Use simultaneous equations to determine the next point of intersection, between equations 2 and 3.

Eqn 1: $y=-\frac{9}{2}x+9$

Eqn 2: $y=-\frac{8}{3}x+8$

$$-\frac{9}{2}x+9=-\frac{8}{3}x+8$$
$$9-8=\left(-\frac{8}{3}+\frac{9}{2}\right)x$$
$$x=\frac{6}{11}$$

By substitution:
$$y=-\frac{9}{2}\left(\frac{6}{11}\right)+9$$
$$y=\frac{72}{11}$$

[1] The original curve or envelope would be tangent to the straight line equations, as the number of lines over the interval is increased successive points of intersection would come closer and closer to the curve.

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Question: 9.

Use CAS to determine the point of intersection between $f(x,3)$ and $f(x,4)$.

$$\left(\frac{12}{11},\frac{56}{11}\right)$$

Question: 10.

Complete the table below for the points of intersection between successive lines.

Question: 11.

Use the difference table to help identify the nature of the pattern in the x coordinates. Based on the results determine an equation in terms of the equation number t.
Note: When t = 1 this will be the point of intersection between equations 1 and 2. When t = 2, this will be the point of intersection between equations 2 and 3. Δ_2 = Constant, therefore quadratic.

Eqn. Nos.	Point of Intersection
1 & 2	$\left(\frac{2}{11},\frac{90}{11}\right)$
2 & 3	$\left(\frac{6}{11},\frac{72}{11}\right)$
3 & 4	$\left(\frac{12}{11},\frac{56}{11}\right)$
4 & 5	$\left(\frac{20}{11},\frac{42}{11}\right)$
5 & 6	$\left(\frac{30}{11},\frac{30}{11}\right)$
6 & 7	$\left(\frac{42}{11},\frac{20}{11}\right)$
7 & 8	$\left(\frac{56}{11},\frac{12}{11}\right)$
8 & 9	$\left(\frac{72}{11},\frac{6}{11}\right)$
9 & 10	$\left(\frac{90}{11},\frac{2}{11}\right)$

x-Coordinate	Δ_1	Δ_2
$\frac{2}{11}$	$\frac{4}{11}$	$\frac{2}{11}$
$\frac{6}{11}$	$\frac{6}{11}$	$\frac{2}{11}$
$\frac{12}{11}$	$\frac{8}{11}$	$\frac{2}{11}$
$\frac{20}{11}$	$\frac{10}{11}$	$\frac{2}{11}$
$\frac{30}{11}$	$\frac{12}{11}$	$\frac{2}{11}$
$\frac{42}{11}$	$\frac{14}{11}$	$\frac{2}{11}$
$\frac{56}{11}$	$\frac{16}{11}$	$\frac{2}{11}$
$\frac{72}{11}$	$\frac{18}{11}$	
$\frac{90}{11}$	**Rule:** $x=\frac{t(t+1)}{11}$	

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Question: 12.

Explain the CAS instruction[2]: $\text{solve}\left(f(x,t)=f(x,t+1),x\right)$

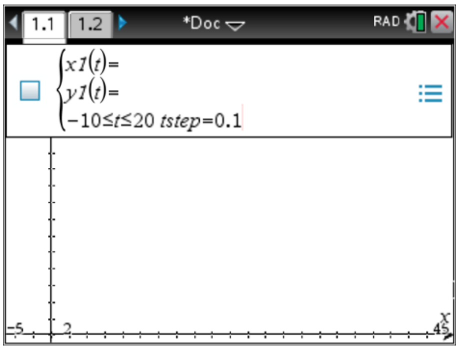
This instruction finds the x-coordinate for the point of intersection between consecutive equations.

$$x=\frac{t(t+1)}{11}$$

Question: 13.

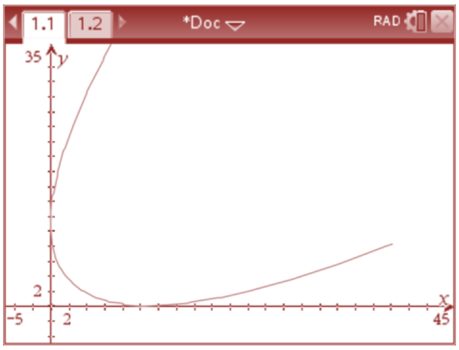
Determine the equation for the y coordinate of the successive points of intersection.

By substitution: $y=\frac{t^2}{11}-\frac{21t}{11}+10$



Question: 14.

On the Graph application, change the graph type to parametric and use the equations from Question 12 for the x coordinate and Question 13 for the y coordinate. Change the step size to 0.1 and the domain for t: $-10 \le t \le 20$. Window settings include x-min = -5, x-max = 50, y-min = -5 and y-max = 35.



Extension - Polynomials

Question: 15.

A polynomial is an expression containing the summation of one or more variables with integer powers and corresponding coefficients.

a. Use the parametric equations to write a polynomial involving x and y for the curve produced by the points of intersection.

$$y=\frac{t^2}{11}-\frac{21t}{11}+10$$

$$y=\frac{t^2+t}{11}-2t+10$$

$$y=\frac{t(t+1)}{11}-2t+10 \quad \text{... by substitution ...}$$

$$y=x-2t+10$$

$$t=\frac{x+10-y}{2}$$

$$x=\frac{t(t+1)}{11}$$

$$x=\frac{(x+10-y)(x+12-y)}{44}$$

$$44x=x^2-2xy+22x+y^2-22y+120$$

$$0=x^2-2xy-22x+y^2-22y+120$$

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[2] To test this command the list t must be deleted. The delete variable command is in the Actions menu: DelVar t

b. Use a selection of appropriate points to show that your equation is an accurate representation of the curve created by the points of intersection of consecutive lines.

Answers will vary.

$t=1 \qquad x=\frac{2}{11} \qquad y=\frac{90}{11}$

Example: $x^2-2xy-22x+y^2-22y+120=$

$$\left(\frac{2}{11}\right)^2-2\left(\frac{2}{11}\right)\left(\frac{90}{11}\right)+\left(\frac{90}{11}\right)^2-22\left(\frac{90}{11}\right)^2+120=0$$

Note that the question states to use "a selection of points" to show that the equation is an "accurate representation of the curve". As the curve was derived from the successive points of intersection the 'points' should come from these points. If students use the 'solve' command for values of x or y they may get two answers for the corresponding x or y value and should therefore explain why.

c. Show that the derived continuous equation it not an accurate representation of the limiting case where infinitely many lines would form a smooth curve.

There are many ways to 'show' that the continuous equation is not an accurate representation. One approach is to consider the x coordinates of points of intersection. Given the nature of the straight line equations it is reasonable to expect that successive points of intersection would contain x coordinates that are greater than zero.

Let $x=0 \qquad x^2-2xy-22x+y^2-22y+120=0$

$(0)^2-2(0)y-22(0)+y^2-22y+120=0$

$y^2-22y+120=0$ … so what if $y = 11$?

$(y-10)(y-12)=0$

$y=10$ or $y=12$

When $y = 11$, $x \approx -.0227$ or $x \approx 44.0227$

The same approach can be used to show that curve crosses the y – axis also. So the curve is a good approximate model, however to obtain a model for the limiting case where points get closer and closer together would require a different approach.

Teacher Notes:

This is activity is part 1 of 3. In this investigation the initial linear equations are simplistic but the curve is somewhat more complicated than for Part 2. In Part 2 the initial equations are slightly harder but the final curve is much easier.

This activity can also be used in Year 12 Specialist Mathematics as the resulting polynomial in the extension questions can be used to compare the gradient using:

- Implicit differentiation
- Chain rule (from the parametric equations)

Students can also investigate better models by exploring solutions to:

$\text{solve}\left(f(x,t)=f(x,t+\lambda),x\right)$ as $\lambda \to 0$. The resultant equation

$f(x,y)=x^2-2xy-22x+y^2-22y+121$ provides a much better model for the curve.

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