A Square Peg in a Round Hole

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Student Activity - Answers

7 8 9 10 **11** **12**

TI-Nspire | Investigation | Student | 45 min

Introduction

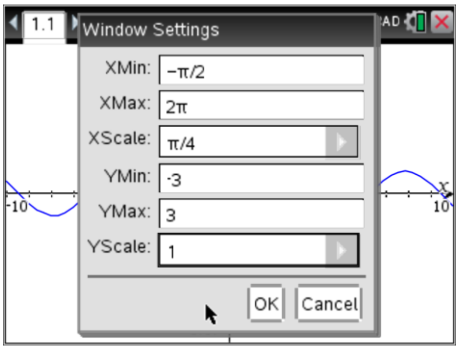
Using trigonometric functions such as $\sin(x)$ on a calculator is relatively easy, enter an angle and the ratio is produced immediately and generally accurate to at least ten decimal places. With the exception of a small group of ratios involving exact values, it is generally only the first four decimal places that are relevant to most calculations. In years gone by, prior to the wide spread use of calculators, trigonometric tables only provided such accuracy. This investigation looks at approximating trigonometric functions with a polynomial, this may seem like trying to fit a square peg in a round whole, but the results can be quite impressive!

A Quadratic Model

Start a new TI-nspire document, insert a Graphs application and graph the function:

$$f(x) = \sin(x)$$

Change the window settings to match those shown. Make sure document settings are set to radians.



Operating system 4.0 onwards provides for exact value specification in the Window Settings. For some earlier operating systems exact values can be achieved by entering them directly on the Graph application.

Question: 1.

Determine exact solutions to $\sin(x) = 0$ over the domain: $0 \le x < 2\pi$. Answer: $x = 0, \pi$

Question: 2.

The quadratic function: $g(x) = (x - m)(x - n)$ passes through the same x intercepts as the function in question 1. Determine the values of *m* and *n*. Answer: $m = 0$ and $n = \pi$

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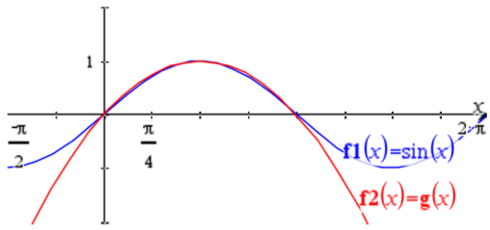
Question: 3.

The quadratic function is further modified: $g(x)=a(x-m)(x-n)$. The function passes through the same x intercepts and also shares the same maximum value as: $\sin(x)$. Determine the exact value of *a* and graph the function: $g(x)$ and $\sin(x)$ on the same set of axis.

$$g\left(\frac{\pi}{2}\right)=1$$

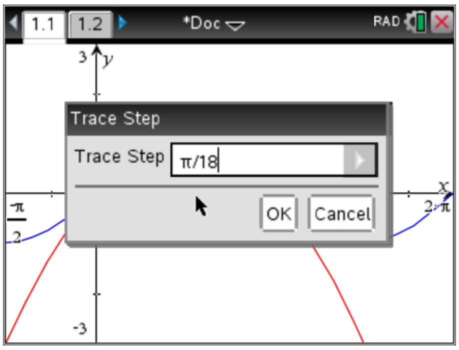
$$a\cdot\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{2}-\pi\right)=1 \quad \text{or} \quad solve\left(g\left(\frac{\pi}{2}\right)=1,a\right)$$

$$a=\frac{-4}{\pi^2}$$



Press the **[menu]** key, select **Trace** followed by **Trace Step**.

Set the trace step to: $\frac{\pi}{18}$.



Return to the trace menu and select **Trace All**.
This will produce a vertical line passing through all the defined functions. Use the right or left arrow to systematically explore the values produced by $\sin(x)$ and $g(x)$.

Question: 4.

Complete the table of values below, correct to TWO decimal places. Comment on the accuracy of $g(x)$ to produce values for: $\sin(x)$.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$	0	0.18	0.34	0.50	0.64	0.77	0.87	0.94	0.99	1.00
$g(x)$	0	0.21	0.40	0.56	0.69	0.80	0.90	0.95	0.99	1.00

Question: 5.

Explain why it is only necessary (and appropriate) to explore the differences over the domain: $0\le x\le\frac{\pi}{2}$.

The periodic nature of sin(x) means that all other values could be generated from these and the symmetry of the quadratic function means it will generate the appropriate values for $\frac{\pi}{2}\le x\le\pi$.

Beyond this overall domain the model is completely the wrong fit.

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Question: 6.

Explain why a cubic function may be less appropriate to model $\sin(x)$ over the domain: $0 \le x \le \pi$ but more appropriate over the domain: $0 \le x \le 2\pi$.

Cubic functions do not contain line symmetry so turning points are not half way between x axis intercepts, compared with sin(x) and quadratic functions where turning points are midway between axis intercepts. However they will provide a slightly better fit over the larger domain due to the general shape of the cubic function. (Additional turning point)

Considering $\sin(x)$ has more than two x intercepts, it follows that increasing the degree of the polynomial is required to provide an improved model.

Consider a quartic function of the form: $h(x) = a(x-m)(x-n)(x-p)(x-q)$ that is used to model $\sin(x)$ over the domain: $-\pi \le x \le 2\pi$.

Question: 7.

Given $h(x)$ and $\sin(x)$ share the same x intercepts over this domain and $h\left(\frac{\pi}{2}\right) = 1$, determine the rule for $h(x)$. x intercepts: $\frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi$ $\quad h(x) = \frac{16}{9\pi^4} x\left(x+\pi\right)\left(x-\pi\right)\left(x-2\pi\right)$

Question: 8.

Complete the table of values below, correct to THREE decimal places. Comment on the accuracy of $h(x)$ to produce values for: $\sin(x)$ and compare this to predictions using: $g(x)$.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$	0	0.174	0.342	0.500	0.643	0.766	0.866	0.940	0.985	1.00
$h(x)$	0	0.191	0.369	0.528	0.668	0.785	0.878	0.945	0.986	1.00

Comparison: The average difference between sin(x) and h(x) is 0.014 and the average difference between sin(x) and g(x) is 0.027 so h(x) is a better approximation.

A Polynomial Model

The $\sin(x)$ function has infinitely many x intercepts, so it follows that a polynomial of 'infinite' degree would be required; the square peg gets bigger!

Consider a polynomial defined as: $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$, rather than modelling $\sin(x)$ using x intercepts, the model will be based on some simple calculus properties of $\sin(x)$.

First derivative: $\frac{d(\sin(x))}{dx} = 1$ when $x = 0$

Second derivative: $\frac{d^2(\sin(x))}{dx^2} = 0$ when $x = 0$

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Third derivative: $\frac{d^3(\sin(x))}{dx^3} = -1$ when $x = 0$

Fourth derivative: $\frac{d^4(\sin(x))}{dx^4} = 0$ when $x = 0$

After the fourth derivative the sequence above cycles, so the fifth derivative becomes 1, the sixth becomes 0, the seventh becomes -1 and the eighth is 0 again. This property can be used to determine a polynomial model for $\sin(x)$ based on $p(x)$ rather than the intercept method with surprising results!

Question: 9.

Given $p(0) = 0$ determine the value of a_0. [Step 1]

Since $x = 0$ all other terms in $p(x)$ other than a_0 also equal zero; therefore $a_0 = 0$

Question: 10.

Given $\frac{d(p(x))}{dx} = 1$ when $x = 0$, determine the value of a_1. [Step 2 – First derivative]

Since $x = 0$ and $p'(x) = a_n nx^{n-1} + ... + 3a_3x^2 + 2a_2x + a_1$ all terms, other than a_1, involved a product that includes zero; therefore $a_1 = 1$

Question: 11.

Given $\frac{d^2(p(x))}{dx^2} = 0$ when $x = 0$, determine the value of a_2. [Step 3 – Second derivative]

Since $x = 0$ and $p''(x) = a_n n(n-1)x^{n-2} + ...6a_3x + 2a_2$ all terms other than a_2, involve a product that includes zero; therefore $a_2 = 0$

Question: 12.

Given $\frac{d^3(p(x))}{dx^3} = -1$ when $x = 0$, determine the value of a_3. [Step 4 – Third derivative]

Since $x = 0$ and $p'''(x) = a_n n(n-1)x^{n-2} + ...6a_3$ all terms other than a_3, involve a product that includes zero; therefore $a_3 = \frac{-1}{6}$. It is also important to note that 6 was produced by 3×2 via repeated differentiation.

Question: 13.

Given $\frac{d^4(p(x))}{dx^4} = 0$ when $x = 0$, determine the value of a_4. [Step 5 – Fourth derivative]

As above... $a_4 = 0$.

Question: 14.

Use the process from the previous four questions to determine values for: a_5, a_6, a_7 and a_8.

$\frac{d^5(p(x))}{dx^5} = 1$ therefore $a_5 = \frac{1}{120}$ or $a_5 = \frac{1}{5!}$; $\frac{d^6(p(x))}{dx^6} = 0$ therefore $a_6 = 0$;

$\frac{d^7(p(x))}{dx^7} = -1$ therefore $a_7 = \frac{-1}{7!}$; $\frac{d^8(p(x))}{dx^8} = 0$ therefore $a_8 = 0$;

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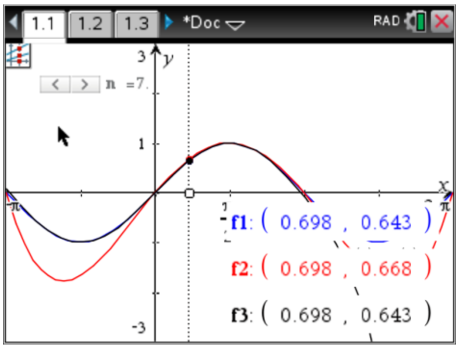
Define the function for $p(x)$ with the computed values for $a_0 \ldots a_8$ and graph this function with $\sin(x)$ over the domain: $-\pi \le x \le 2\pi$.



A sample trace of the graphs is shown opposite. The graph of $f_1(x) = \sin(x)$, $f_2(x)$ is the quartic function and $f_3(x)$ is the new polynomial, notice the accuracy at this particular point?

Question: 15.

Define $p(x)$ as a polynomial of degree 7 and explain why, based on the previous calculations for a_8, it is not necessary to use a polynomial of degree 8.

$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ Since $a_8 = 0$ the highest power of x is 7 so $p(x)$ is a polynomial degree 7.

Question: 16.

Based on the patterns established so far, predict values for: a_9, a_{10}, a_{11} and a_{12}.

$a_9 = \frac{1}{9!}$, $a_{10} = 0$, $a_{11} = \frac{-1}{11!}$ and $a_{12} = 0$

Question: 17.

Complete the table of values below, correct to FOUR decimal places. Comment on the accuracy of $p(x)$ to produce values for: $\sin(x)$.

Within the graph application, use the [**menu**] to change the **settings** for the graph application. Change the display digits to 4 decimal places.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$	0	0.1745	0.3420	0.5000	0.6428	0.7660	0.8660	0.9397	0.9848	1.0000
$p(x)$	0	0.1745	0.3420	0.5000	0.6428	0.7660	0.8660	0.9397	0.9848	0.9998

The polynomial predicts values for sin(x) accurate to 4 decimal places, with the exception of $\pi/2$.

Question: 18.

Given $\frac{d(\sin(x))}{dx} = \cos(x)$, determine a polynomial $q(x)$ for $\cos(x)$ and graph this function against $\cos(x)$. Comment on the accuracy of this polynomial.

Depending on the degree used for $p(x)$ then $q(x) = \frac{-x^{10}}{10!} + \frac{x^8}{8!} - \frac{x^6}{6!} + \frac{x^4}{4!} - \frac{x^2}{2!} + 1$. Even using $q(x)$ as a polynomial of degree 6, computed values are generally accurate to 4 decimal places.

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Question: 19. Define a rational function for $\tan(x)$ and determine the accuracy of this function.

$$\tan(x) \approx \frac{\frac{-x^7}{7!} + \frac{x^5}{5!} - \frac{x^3}{3!} + x}{\frac{x^8}{8!} - \frac{x^6}{6!} + \frac{x^4}{4!} - \frac{x^2}{2!} + 1} = \frac{-8x^7 + 336x^5 - 6720x^3 + 40320x}{x^8 - 56x^6 + 1680x^4 - 20160x^2 + 40320}$$

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