# Positive and Negative Angles and Arcs

## Math Objectives

- Students will understand that if the intersection point *P* of two lines lies inside a circle, then the measure of the angle formed by the two secants is equal to the average of the measures of the arcs intercepted by that angle and its corresponding vertical angle.
- Students will learn that an oriented angle is defined to be the measure of rotation of a ray about a common point (the vertex).
- Students will learn that an oriented arc's measure will be equal to the measure of its oriented central angle. The arc's measure will be positive or negative depending on the angle's orientation.
- Students will determine that if two lines intersect each other and also intersect a circle, then the measure of an angle of intersection of the two lines is equal to the average of the measures of the angle's intercepted arcs.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

## Vocabulary

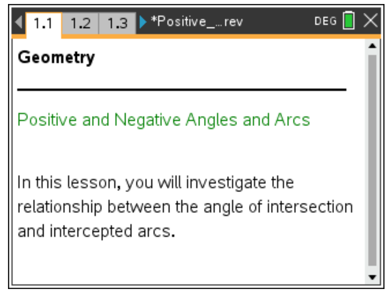
- secant and chord
- tangent line
- central angle
- intercepted arc
- oriented angle
- initial ray and terminal ray
- reflex angle

## About the Lesson

- This lesson involves measuring oriented angles and applying these measurements to reflex angles and arcs.
- As a result students will:
  - Observe the relationship between the measure of the angle of intersection of two lines and the measures of the intercepted arcs.
  - Deduce that the angle of the intersection has a measure equal to the average of the measures of the intercepted arcs.

## TI-Nspire™ Navigator™ System

- Quick Poll
- Screen Capture




**TI-Nspire™ Technology Skills:**

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

**Tech Tips:**

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- You can hide the function entry line by pressing ctrl G.

**Lesson Materials:**

*Student Activity*

Postive_and_Negative_Angles_and_Arcs_Student.pdf

Postive_and_Negative_Angles_and_Arcs_Student.doc

*TI-Nspire document*

Postive_and_Negative_Angles_and_Arcs.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

### Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging the point, check to make sure that they have moved the arrow until it becomes a hand ( ). Press ctrl [click] to grab the point and close the hand ( ). After the point has been moved, press esc to release the point.

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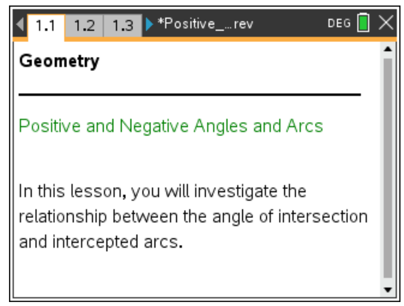
This activity begins with an interactive exploration with oriented angles. Then you will explore several cases in which two lines intersect at a point and also intersect a circle. You will investigate the relationship between the angle of intersection and the intercepted arcs.

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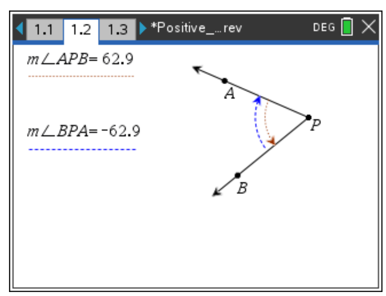
An oriented angle is defined to be the measure of rotation of a ray about a common point (the vertex). One side of the angle is the initial ray and the other is the terminal ray, which is the result of a rotation of a given number of degrees, either in the clockwise or counterclockwise direction.

**Move to page 1.2.**




$\angle APB$ and $\angle BPA$ are the same angle but have different orientations. Because A is listed first in the labeling of $\angle APB$, $\overrightarrow{PA}$ is the initial ray and $\overrightarrow{PB}$ is the terminal ray. In the case of $\angle BPA$, $\overrightarrow{PB}$ is the initial ray and $\overrightarrow{PA}$ is the terminal ray.

If the initial ray is rotated in a counterclockwise direction toward the terminal ray, then its measure is positive. If the initial ray is rotated in the clockwise direction, then its measure is negative.

1. Drag point *A* counterclockwise past point *B*. What changes occurred in the angle measures?

 **<u>Answer:</u>** The measure of $\angle APB$ became negative and the measure of $\angle BPA$ is positive.

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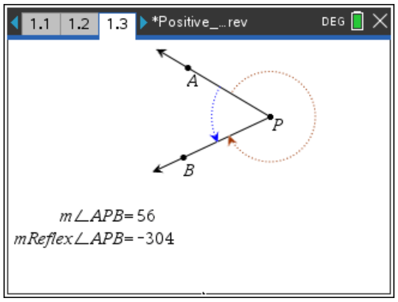


2. The measure $\angle APB$ is shown as well as the measure of its reflex angle. A reflex angle is one that involves a rotation of more than $180°$ but less than $360°$.

   a. What do you notice about the relationship of the measures of these angles? How does it relate to the definition of an oriented angle?

   **Answer:** One of the angles has a positive measure and the other has a negative measure. This is because for $\angle APB$, you must rotate $\overrightarrow{PA}$ counterclockwise to reach $\overrightarrow{PB}$. But for reflex $\angle APB$ you have to rotate $\overrightarrow{PA}$ clockwise to reach $\overrightarrow{PB}$. If you take the absolute value of the measures, their sum is 360°.

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| **TI-Nspire Navigator Opportunity: *Screen Capture***<br>**See Note 1 at the end of this lesson.** |

   b. Drag point *A* counterclockwise along the circle past point *B*. What changes occurred in the angle measures? How do these changes relate to the definitions for an oriented angle and a reflex angle?

   **Answer:** The measures of the angles change signs because the initial ray now has to be rotated in the opposite direction for both the oriented central angle and the oriented reflex angle.

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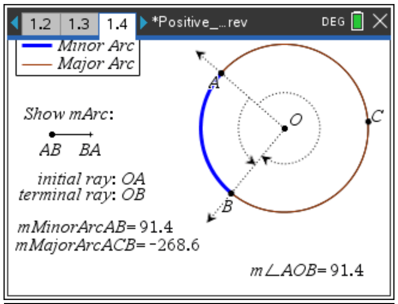


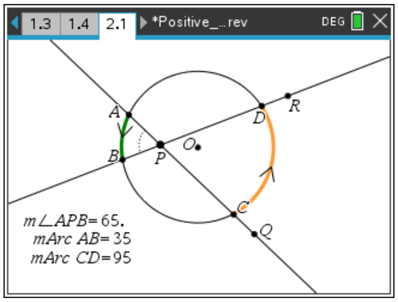

Both the minor arc *AB* and the major arc *ACB* are indicated on the circle. The measures of these arcs are also shown. Recall that the measure of an arc is equal to the measure of its central angle. The measure of $\angle AOB$ is also shown.

On this page, there is slider under the words *Show mArc:*. Notice that when the slider is set at *AB*, $\overrightarrow{OA}$ is the initial ray and the measures of both the minor arc *AB* and the major arc *ACB* are shown. Drag the point on the slider from *AB* to *BA*. Notice that $\overrightarrow{OB}$ is now the initial ray and the measures of minor arc *BA* and major arc *BCA* are shown.

3. Move point *A* around the circle with the slider in different positions. What changes do you see in the measures of these arcs? How do these changes relate to the definition of an oriented angle?

   **Answer:** The measure of the minor arc will always equal the measure of the oriented central angle. The measure of the major arc will always equal the measure of the oriented reflex angle.

**Move to page 2.1.**




The measures of $\angle APB$, arcs *AB* and *CD* are shown. The arrows on arcs *AB* and *CD* indicate their orientation.

4. Drag point *P* around in the interior of the circle, but not at the center of the circle.
   a. With respect to the circle, what type of lines are $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$?

      **Answer:** secant lines

   b. What relationship do you notice between the measures of the arcs and the measure of $\angle APB$?

      **Answer:** Half of the sum of the measures of the arcs equals the measure of $\angle APB$. (Or, the average of the measures of the arcs equals the measure of $\angle APB$.)

**TI-Nspire Navigator Opportunity: *Screen Capture***
**See Note 2 at the end of this lesson.**

5. Drag point *P* to coincide with the center.
   a. With respect to the circle, what type of lines are $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$?

      **Answer:** secant lines that contain diameters

   b. Does the relationship you found in question 4b still hold? If the relationship you found does not hold, revise the relationship you stated so that it is true wherever *P* is in the interior.

      **Answer:** The relationship still holds, but in this case each of the arcs is also equal to the measure of $\angle APB$, since it is now a central angle.

6. Drag points *R* or *Q* so that the orientation of arc *CD* changes. Drag point *P* anywhere in the interior of the circle. Does the relationship you have found still hold? If not, revise the relationship so that it is true wherever *P* is in the interior of a circle and for both positive and negative arc measures.

   **Answer:** The relationship still holds because now the arcs and the angle have negative measures. Half of the sum of the measures of the two arcs (or the average of the arc measures) still equals the measure of $\angle APB$.

7. Drag point *P* around the exterior of the circle, but make sure $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ both intersect the circle in two points. Does the relationship you found still hold? If the relationship you found does not hold, revise the relationship so that it is true regardless of whether *P* lies in the interior or exterior of the circle.

   **Answer:** The relationship found in question 4b is still true. That is the angle measures the same as one-half the sum of the measures of the arcs (or the average of the arc measures).

8. Leave *P* in the exterior of the circle and $\overleftrightarrow{PR}$ intersecting the circle in two points. Drag point *Q* so that point *C* coincides with point *A*.
   a. With respect to the circle, what type of lines are $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$?

      **Answer:** $\overleftrightarrow{PA}$ is a tangent line and $\overleftrightarrow{PB}$ is a secant line.

   b. Is the relationship you found still true? If the relationship you found is not true, revise it for this case.

      **Answer:** The relationship is still true.

9. Leave *P* in the exterior of the circle, and leave *C* so that it still coincides with point *A*. Drag point *R* so that point *B* coincides with point *D*.
   a. With respect to the circle, what type of lines are $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$?

      **Answer:** They are both tangent lines.

   b. Let *CD* denote the major arc and *AB* the minor arc. Is the relationship you found still true? If the relationship you found is not true, revise it for this case.

      **Answer:** The relationship is still true.

10. Bryan says that if two secants intersect in the interior of a circle, then the measure of an angle formed is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle. Which of questions 4–9, if any, support his conjecture?

**Answer:** This is true. Questions 4, 5, and 6 support his conjecture.

11. Dajah says that if two secants intersect in the exterior of a circle, then the measure of the angle formed is one-half the sum of the measures of its intercepted arcs. Which of questions 4–9, if any, support her conjecture?

**Answer:** This is true. Question 7 supports her conjecture.

12. Michael agrees with Dajah, but he also thinks her statement is true for the case when there is a secant and a tangent, and also for the case when there are two tangents. Which of questions 4–9, if any, support his conjecture?

**Answer:** This is true, and is supported by questions 8 and 9.

13. Chloe agrees with Bryan and Dajah, but believes that she has an idea that works for all of the cases. She thinks that if two lines intersect each other and also intersect a circle, then the measure of an angle of intersection of the two lines is the average of the angle's intercepted arcs. Which of questions 4–9, if any, support her conjecture?

**Answer:** This is true. The results from questions 4–9 support her conjecture.

**TI-Nspire Navigator Opportunity: *Quick Poll***
**See Note 3 at the end of this lesson.**

14. Alejandro says the following formula works for questions 4–9. Do you agree or disagree? Why?

$$\angle APB = \frac{mArc\ AB + mArc\ CD}{2}$$

**Answer:** This is true because one-half the sum will always equal the measure of the angle.

## Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- When considering oriented angles and oriented arcs, no matter where *P* lies and no matter whether you have secants or tangents, the following equation holds:

$$\angle APB = \frac{mArc\ AB + mArc\ CD}{2}$$

## TI-Nspire Navigator

**Note 1**

**Question 2a, *Screen Capture:*** Take a *Screen Capture* so that students can use each other's screens to discuss the relationship of the measures of the angles.

**Note 2**

**Question 4b, *Screen Capture:*** If students are having difficulty recognizing a relationship between the angle and arc measures, take a *Screen Capture* so that students can use each other's screens to discuss the relationship.

**Note 3**

**Questions 10–13, *Quick Poll:*** Have students send in their responses to the question numbers that support each conjecture through an *Open Response Quick Poll*.