Induction for Tetrahedral Numbers

Teacher Notes and Answers

7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 50 min

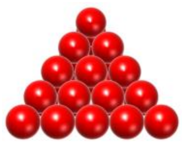
Teacher Notes:

"Inductive Tetrahedrals" is part two of this three-part series. [Refer to Inductive Whole Numbers for Part One].
Part two follows a similar format to part one but is slightly more challenging from an algebraic perspective. Teachers can use part two in this series as a more independent environment for students focusing on providing feedback rather than instruction. A set of Power Point slides is provided that can be used to enhance the visual aspect of this lesson, the ability to visualise solutions can give problems more meaning.

"Cubes and Squares" is part three of this three-part series. This section is designed as an assessment tool, marks have been allocated to each question. The questions are more reflective of the types of questions that students might experience in the short answer section of an examination where calculators may be used to derive answers or to simply verify by-hand solutions. A set of Power Point slides is provided that can be used to introduce students to this task.

Introduction

The purpose of this activity is to use exploration and observation to establish a rule for the sum of the first *n* tetrahedral numbers then use proof by mathematical induction to show that the rule is true for all whole numbers. What are the Tetrahedral numbers? The prefix 'tetra' refers to the quantity four, so it is not surprising that a tetrahedron consists of four faces, each face is a triangle. This triangular formation can sometimes be found in stacks of objects. The series of diagrams below shows the progression from one layer to the next for a stack of spheres.

Row Number	1	2	3	4	5
Items Added					
Complete Stack					

Question: 1.

Create a table of values for the row number and the corresponding quantity of items that are added to the stack.

Row Number	1	2	3	4	5
Items Added	1	3	6	10	15

Question: 2.

Create a table of values for the row number and the corresponding quantity of items in a complete stack.

Row Number	1	2	3	4	5
Complete Stack	1	4	10	20	35

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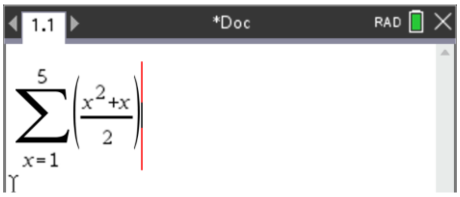
Question: 3.

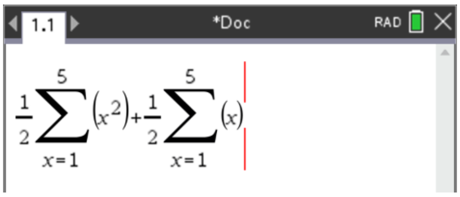
The calculator screen shot shown here illustrates how to determine the fifth tetrahedral number. The same command could be used to determine any of the tetrahedral numbers.
Explain how this command is working.



Answer: The sum of first *n* whole numbers = n x (n + 1) ÷ 2 (triangular numbers) is the expression in the summation (sigma) command. From the diagrams, it can be seen that each time another row is 'added' to the stack, to create the next tetrahedral number, the quantity added is a triangular number. The tetrahedral numbers are the cumulative sum of the triangular numbers.

Question: 4.

Verify that the calculation shown opposite is the same as the one generated in Question 3.



Answer: Students may look at the general case, this involves an extraction of the common factor (1/2) [Application of the distributive law]. The original expression: $(x^2 + x)$ has simply been split into two separate calculations. [Application of the commutative law for addition.].

In the case of a single calculation, students may simply note that:

$\sum_{x=1}^{5}\left(\frac{x^2+x}{2}\right) = 35$ compared with $\frac{1}{2}\sum_{x=1}^{5} x^2 = \frac{55}{2}$ and $\frac{1}{2}\sum_{x=1}^{5} x = \frac{15}{2}$ which also totals to 35.

Question: 5.

Enter the numbers 1, 2 … 10 in List 1 on the calculator. Enter the first 10 tetrahedral numbers in List 2. Once the values have been entered try the following:

a) Quadratic regression using List 1 and List 2. Check the validity of the result via substitution.

Answer: As the relationship is not quadratic, the quadratic regression does not predict the values for the tetrahedral numbers. Equation: $3.25x^2 - 12.35x + 14.3$. Students should see instantly that this equation is not going to predict values correctly. Substituting x = 0, x = 1 … produce erroneous results.

b) Cubic regression using List 1 and List 2. Check the validity of the result via substitution.

Answer: Cubic regression seems to do a perfect job of a formula:

Equation: $\frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$ which can also be represented as: $\frac{x^3+3x^2+2x}{6} = \frac{x(x+1)(x+2)}{6}$. Students can use direct substitution or define a function on the calculator and see that f(1)=1, f(2)=4, f(3)=10 … f(10)=220.

Pascal's Triangle – Another Gem

Pascal's triangle also contains the tetrahedral numbers.

Example: The number 20 is the 4th tetrahedral number. It is located in the 6th row.

Recall that the elements in Pascal's triangle can be computed using combinatorics: ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

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Question: 6.

Use combinatorics to determine the value of the 100th Tetrahedral number. Check your answer using the equation established in the previous question and the summation tool on the calculator.

Answer:

Using combinatorics: ${}^{102}C_3 = 171700$.

Using the cubic equation from Question 5: $\dfrac{x(x+1)(x+2)}{6} = \dfrac{100\times101\times102}{6} = 171700$

Using the calculator's summation command: $\displaystyle\sum_{x=1}^{100}\frac{x(x+1)}{2} = 171700$

Question: 7.

Use Pascal's triangle to determine a formula for the Tetrahedral numbers.

Answer:

$$\begin{aligned}{}^{n+2}C_3 &= \frac{(n+2)(n+1)n(n-1)(n-2)...}{3!(n-1)(n-2)...}\\ &= \frac{n(n+1)(n+2)}{6}\end{aligned}$$

[Same equation established using cubic regression]

Question: 8.

Given that the Tetrahedral numbers can be computed using:

$$\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right)$$

Use mathematical induction to prove that this is equal to: $\dfrac{x(x+1)(x+2)}{6}$.

Answer

Step 1: Show true for $x = 1$

We must first prove that the formula is true for $x = 1$.

RHS: $\dfrac{x(x+1)(x+2)}{6} = \dfrac{1\times2\times3}{6} = 1$

LHS: $\displaystyle\sum_{n=1}^{1}\left(\frac{n^2+n}{2}\right) = 1$. Therefore LHS = RHS

Step 2: Assume true for x

That is: $\displaystyle\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right) = \frac{x(x+1)(x+2)}{6}$ **-- Equation 1**

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Step 3: Show true for $x + 1$.

Working with the LHS

LHS = $\sum_{n=1}^{x+1}\left(\frac{n^2+n}{2}\right)=\sum_{n=1}^{x}\left(\frac{n^2+n}{2}\right)+\frac{(x+1)^2+(x+1)}{2}$

From Equation 1 we can re-write this as: $\frac{x(x+1)(x+2)}{6}+\frac{(x+1)^2+(x+1)}{2}$

Common denominator: $\frac{x(x+1)(x+2)}{6}+\frac{3(x+1)^2+3(x+1)}{6}$

Common factor: $\frac{(x+1)\left[x(x+2)+3(x+1)+3\right]}{6}$

This simplifies to: $\frac{(x+1)(x^2+5x+6)}{6}$

Factorising: $\frac{(x+1)(x+2)(x+3)}{6}$

Working with the RHS:

Replace x with $x+1$ so RHS becomes: $\frac{(x+1)(x+2)(x+3)}{6}$

$\therefore$ LHS = RHS

Question: 9.

Question 4 used the property that $\sum_{n=1}^{x}\frac{n^2+n}{2}=\frac{1}{2}\sum_{n=1}^{x}n^2+\frac{1}{2}\sum_{n=1}^{x}n$.

Use this to show that: $\sum_{n=1}^{x}n^2=\frac{x(x+1)(2x+1)}{6}$

Answer

$$\begin{aligned}\sum_{n=1}^{x}n^2&=2\sum_{n=1}^{x}\frac{n^2+n}{2}-\sum_{n=1}^{x}n\\&=\frac{2x(x+1)(x+2)}{6}-\frac{x^2+x}{2}\\&=\frac{2x^3+6x^2+4x}{6}-\frac{3x^2+3x}{6}\\&=\frac{2x^3+3x^2+x}{6}\\&=\frac{x(x+1)(2x+1)}{6}\end{aligned}$$

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Question: 10.

Use mathematical induction to prove that: $\sum_{n=1}^{x} n^2 = \frac{x(x+1)(2x+1)}{6}$

Answer

Step 1: Show true for $x = 1$

LHS: $\sum_{n=1}^{1}\left(n^2\right)=1$

RHS: $\frac{1\times(1+1)\times(2+1)}{6}=1$. $\therefore$ LHS = RHS

Step 2: Assume true for x

That is: $\sum_{n=1}^{x}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}$ **-- Equation 1**

Step 3: Show true for x + 1.

Working with the LHS

$$\sum_{n=1}^{x+1}\left(n^2\right)=\sum_{n=1}^{x}\left(n^2\right)+(x+1)^2$$

$$\sum_{n=1}^{x+1}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}+(x+1)^2$$ -- From Equation 1:

$$\sum_{n=1}^{x+1}\left(n^2\right)=\frac{x(x+1)(2x+1)}{6}+\frac{6(x+1)^2}{6}$$

$$=\frac{(x+1)(2x^2+x+6x+6)}{6}$$

$$=\frac{(x+1)(x+2)(2x+3)}{6}$$

Working with the RHS

$\frac{x(x+1)(2x+1)}{6}$ replace x with $x+1$

$\frac{(x+1)(x+2)(2x+3)}{6}$ $\therefore$ LHS = RHS

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