Induction for Whole Numbers

Teacher Notes and Answers

7 8 9 10 **11** 12

TI-84PlusCE™ Investigation Student 50 min

Teacher Notes:

"Induction for Whole Numbers" is part one of a three-part series. Teachers are encouraged to use this first activity to build student capacity and confidence relating to appropriate calculator use and also the appropriate level of detail required for proof by induction. A set of Power Point slides has been provided that can be used to enhance the visual aspect of this lesson, the ability to visualise solutions can give problems more meaning. The visuals also help explain why the sum of the first n whole numbers are also referred to as 'triangular numbers'.

"Inductive Tetrahedrals" is part two of this three-part series. Part two follows a similar format to part one but is slightly more challenging from an algebraic perspective. Teachers can use part two in this series as a more independent environment for students focusing on providing feedback rather than instruction. A set of Power Point slides is provided that can be used to enhance the visual aspect of this lesson, the ability to visualise solutions can give problems more meaning.

"Cubes and Squares" is part three of this three-part series. This section is designed as an assessment tool, marks have been allocated to each question. The questions are more reflective of the types of questions that students might experience in the short answer section of an examination where calculators may be used to derive answers or to simply verify by-hand solutions. A set of Power Point slides is provided that can be used to introduce students to this task.

Introduction

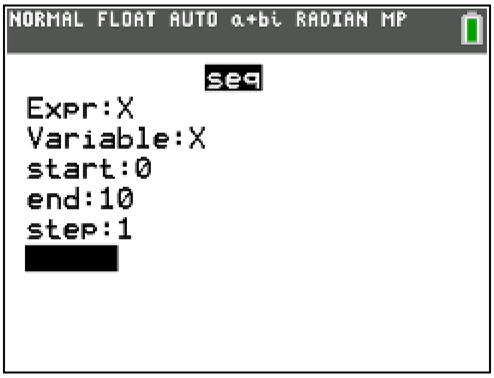
The purpose of this activity is to use exploration and observation to establish a rule for the sum of the first n whole numbers then use proof by induction to show that the rule is true for all whole numbers.

Calculator Instructions

The sequence command can be used to generate the first 10 whole numbers.

These values could be entered directly into a list, however it is often handy to know where and how to use some of the calculator's commands so that when longer lists need to be generated they don't have to be entered individually.

[2nd] [STAT] List > OPS > SEQ



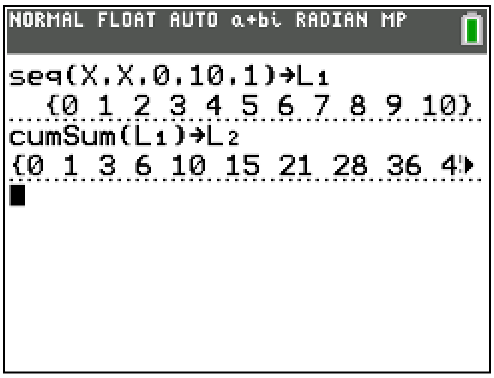
Populate the sequence template as shown opposite and paste into the calculator's home screen.

Store the generated list into L_1.

The cumulative sum of these numbers can also be computed.

[2nd] [STAT] List > OPS > CumSum

Store this cumulative sum in L_2.



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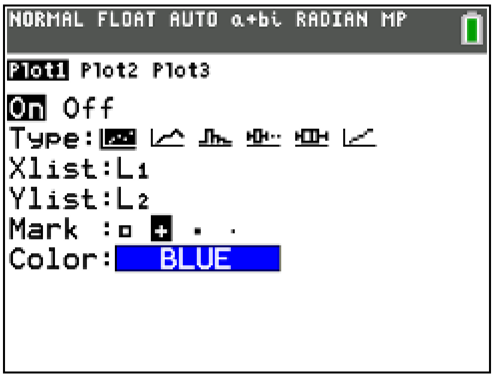
Set up a scatter plot to graph the points where L_1 is plotted on the x axis and L_2 on the y axis.

[2nd] [Y=] Stat Plot > Plot 1

Make sure no other graphs are being plotted; then zoom in on the data.

Zoom > ZoomStat



Question: 1.

With reference to a difference table, explain why the relationship must be quadratic.

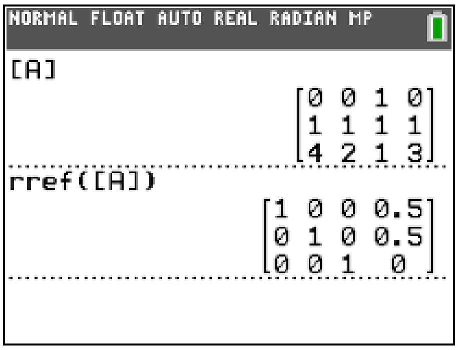
The values in L_2 represent the progressive (cumulative) sum, therefore each value differs by a simple arithmetic sequence: 1, 2, 3, 4 ... each of these values differ by a constant amount, therefore the second order difference is constant resulting in a second degree (quadratic) polynomial.

Teacher Notes:

A difference table can be set up using the lists and the ∆List command to show students that the second order differences are constant, therefore the relationship is a polynomial of degree 2.

Question: 2.

Use simultaneous equations to establish values for a, b and c where the sum (s) can be expressed in the form: $s = ax^2 + bx + c$.



Answers will vary depending on the selection of points. The simplest points would be (0, 0); (1, 1) & (2, 3)

$$Eqn1: \quad 0 = a(0)^2 + b(0) + c$$
$$Eqn2: \quad 1 = a(1)^2 + b(1) + c$$
$$Eqn3: \quad 3 = a(4) + b(2) + c$$

Equation 1 results in c = 0, combining Equations 2 & 3 produces: a = ½ and b = ½.

Teacher Notes:

This set of simultaneous equations are very straight forward, however some students may use matrices or the Polysmlt Application to solve these equations. With regards to matrices, Row Reduction Echelon Form (RREF) is particularly efficient.

The output shows that a = ½, b = ½ and c = 0.

Question: 3.

Graph your equation to check that it passes through the points that have been plotted. Use $Y_1(x)$ and substitute a range of values to check your answer.

Answers will vary. Students can provide a 'screen shot' from their calculator of the graph and use of Y_1 verifying their solutions.

Teacher Notes:

This purpose of this question is to encourage students to use their calculator to check or verify solutions. Many exam questions are written in such a way that the calculator may not be particularly useful in determining a solution. This may be the way the question is written: "Use calculus to determine …the corresponding values of a, b and c.". However, students should familiar with how they can use their calculator to verify their solutions.

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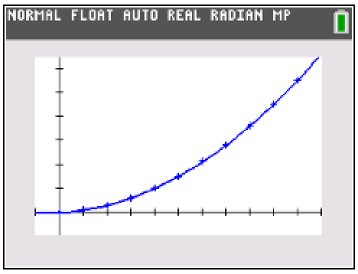
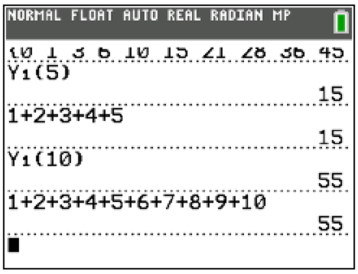
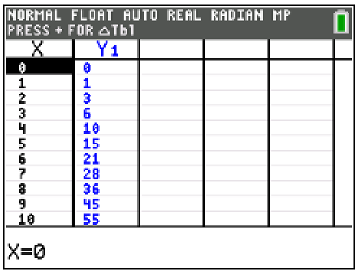


A graph is a quick visual inspection.



Students should be familiar with the use of $Y_1(x)$, similar to f(x).



A table of values provides a quick numerical confirmation.

Question: 4.

Use your formula to determine the sum of the first 50 whole numbers.

Answer: $\frac{50^2}{2}+\frac{50}{2}=1275$

Question: 5.

There is a summation command on the calculator: **Math** > **Summation** or **Alpha** > **F2** > **Summation**. Use this template to determine the sum of the first 100 whole numbers and compare the result with your equation.

Notation: $\sum_{n=1}^{100} n$.

Answer: 5050 and $\frac{100^2}{2}+\frac{100}{2}=5050$ Answers are the same.

Pascal's + Triangle = Hidden Gem

The sum of the first n whole numbers is also referred to as Triangular Numbers. It is a lovely synchronism that Pascal's Triangle contain the triangular numbers.

The n^{th} triangular number is the third element in the $(n+1)^{th}$ row[1].

Example: The number 15 is the 5^{th} triangular number, it is the third element in the 6^{th} row.

Recall that the elements in Pascal's triangle can be computed using combinatorics: ${}^nC_r=\frac{n!}{(n-r)!r!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Generate the numbers: {2, 3, 4 … 12} and store the results in L_1.

A list of values can be generated from the combinatorics command, example: ${}^{L_1}C_2$.

Use this to generate the triangular numbers.

[1] Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

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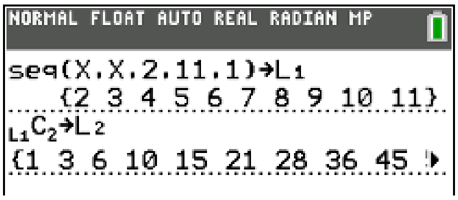
Question: 6.

a) Use your calculator and the combinatorics command to write down the first 10 triangular numbers. [Store the values in L_2]



Answer: ${}^{2}C_2 = 1,\ {}^{3}C_2 = 3,\ {}^{4}C_2 = 6 \ldots {}^{11}C_2 = 55$

b) Use combinatorics to calculate the 100th triangular number, the sum of the first 100 whole numbers.

Answer: ${}^{101}C_2 = 5050$ or $\dfrac{101!}{99!2!} = \dfrac{101\times 100}{2} = 5050$

c) The diagonal for the triangular numbers in Pascal's Triangle can be written using combinatorics:

$${}^{n+1}C_2 = \frac{(n+1)!}{((n+1)-2)!2!}$$

Simplify this formula to write an expression for the n^{th} triangular number.

Answer:

$$\begin{aligned} {}^{n+1}C_2 &= \frac{(n+1)!}{(n-1)!2!} \\ &= \frac{(n+1)(n)(n-1)(n-2)\ldots}{2!(n-1)(n-2)\ldots} \\ &= \frac{(n+1)(n)}{2} \\ &= \frac{n^2+n}{2} \end{aligned}$$

Question: 7.

Use induction (outlined below) to prove that $\displaystyle\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$ is true for all positive whole numbers.

a) Show that your rule is true for x = 1

LHS: $\displaystyle\sum_{n=1}^{1} n = 1$ (Sum of first '1' whole numbers) RHS: $\dfrac{x(x+1)}{2} = \dfrac{1(1+1)}{2} = 1$

b) Assume the result for $\displaystyle\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$ is true and show the rule holds for x + 1

Suggested answer:

LHS:

$$\begin{aligned} \sum_{n=1}^{x+1} n &= \left(\sum_{n=1}^{x} n\right) + x + 1 \\ &= \frac{x(x+1)}{2} + x + 1 \\ &= \frac{x(x+1)}{2} + \frac{2(x+1)}{2} \\ &= \frac{(x+2)(x+1)}{2} \end{aligned}$$

RHS:

$$\begin{aligned} &\frac{x(x+1)}{2} \,|\, \text{substitute: } x+1 \\ &= \frac{(x+1)(x+1+1)}{2} \\ &= \frac{(x+1)(x+2)}{2} \\ &= \frac{(x+2)(x+1)}{2} \end{aligned}$$

$\therefore$ LHS = RHS

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Question: 8.

The sum of the first x odd numbers can be expressed as: $\sum_{n=1}^{x}(2n-1)=x^2$.

a) Use your calculator to check that this equation is true for the first 10 odd numbers.

Answer: Students may simply add: 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100 or they may use the summation command, however they must also state that the RHS generates the same, that is: $10^2 = 100$

b) Use proof by mathematical induction to show that this rule is true for all odd numbers.

Answer:

Show true for x = 1

LHS: $\sum_{n=1}^{1}(2n-1)=1$ [First '1' odd number] RHS: $x^2=(1)^2=1$

Assume true: $\sum_{n=1}^{x}(2n-1)=x^2$

Show true for x + 1:

LHS: $\sum_{n=1}^{x+1}(2n-1)=\sum_{n=1}^{x}(2n-1)+2(x+1)-1$

$=x^2+2x+2-1$

RHS: $x^2 \mid \textit{substitute } x+1$

$(x+1)^2=x^2+2x+1$

$\therefore$ LHS = RHS

Question: 9.

The sum of the first x even numbers can be expressed as: $\sum_{n=1}^{x}(2n)=x^2+x$.

Use proof by mathematical induction to show that this rule is true for all even numbers.

Answer:

Show true for x = 1

LHS: $\sum_{n=1}^{1}(2n)=2$ [First even number] RHS: $x^2+x=(1)^2+1=2$

Assume true: $\sum_{n=1}^{x}(2n)=x^2+x$

Show true for x + 1:

LHS: $\sum_{n=1}^{x+1}(2n)=\sum_{n=1}^{x}(2n)+2(x+1)$

$=x^2+x+2x+2$

$=x^2+3x+2$

RHS: $x^2+x \mid \textit{substitute } x+1$

$(x+1)^2+(x+1)=x^2+2x+1+x+1$

$=x^2+3x+2$

$\therefore$ LHS = RHS

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Question: 10.

Use your calculator to determine a rule for the sum of the multiples of three: {3, 6, 9, 12 …}, then prove your result by mathematical induction. [Hint: Question 9 was a rule for even numbers, multiples of 2]

Answer: Student approaches may vary for the determination of the equation. Generating the lists of numbers and corresponding scatterplot is an efficient approach, followed by quadratic regression to determine the equation:

$$\sum_{n=1}^{x}(3n)=\frac{3}{2}\left(x^2+x\right)$$

Proof by mathematical induction:

Show true for $x = 1$

LHS: $\sum_{n=1}^{1}(3n)=3$ [First even number] RHS: $\frac{3}{2}\left(1^2+1\right)=\frac{3}{2}\times 2=3$

Assume true: $\sum_{n=1}^{x}(3n)=\frac{3}{2}\left(x^2+x\right)$

Show true for $x + 1$:

LHS:

$$\begin{aligned}\sum_{n=1}^{x+1}(3n)&=\sum_{n=1}^{x}(3n)+3(x+1)\\&=\frac{3}{2}\left(x^2+x\right)+3x+3\\&=\frac{3}{2}\left(x^2+x\right)+\frac{3}{2}(2x+2)\\&=\frac{3}{2}(x^2+3x+2)\end{aligned}$$

RHS:

$$\begin{aligned}&\frac{3}{2}\left(x^2+x\right)|\ \textit{substitute}\ x+1\\&\frac{3}{2}\left((x+1)^2+(x+1)\right)=\frac{3}{2}\left(x^2+2x+1+x+1\right)\\&\qquad=\frac{3}{2}\left(x^2+3x+2\right)\end{aligned}$$

$\therefore$ LHS = RHS

Teacher Notes:

Question 10 illustrates how students can use the calculator to determine a formula and then go on to prove the rule they have established. Students should also be encouraged to compare the result for the sum of the first n whole numbers, even number (multiples of 2) and finally the multiples of 3.

The general result: $\sum_{n=1}^{x}(k\times n)=k\sum_{n=1}^{x}n$

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