

# Vector Graphics

## Teacher Notes and Answers

7 8 9 10 11 12



### Introduction

In 1637 René Descartes published a revolutionary book *La Géométrie* where he brought together two areas of mathematics; Geometry and Algebra. The “Cartesian” plane, named after René Descartes, is an integral component of every high school mathematics course. In this activity you will see how vectors can be incorporated into this two dimensional space. Using the same techniques, you can easily extend this approach into three dimensions.

Vectors can be expressed as a direction and magnitude (Polar) or as mutually perpendicular projections (Components), the latter form is explored in this activity. The directions are referred to as  $i$  ( $x$  direction) and  $j$  ( $y$  direction), for three dimensions a  $k$  ( $z$  direction) is also included. For some relatively straight forward problems it is quicker and more appropriate to complete calculations ‘by-hand’, however, as the calculations are repeated and gradually increase in complexity, having a template becomes an extremely useful tool. This activity sets up a template that can be used for other investigations including problems in three dimensions.

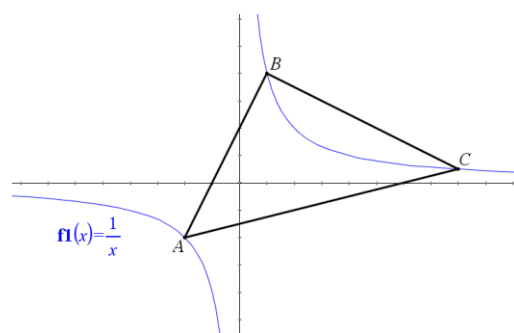
### Problem to be explored

Consider three points A, B and C on the function:  $y = \frac{1}{x}$ .

These three points can be joined to create a triangle:  $\triangle ABC$ .

This triangle has some interesting features that will be explored through the use of vectors.

To begin this investigation, point A =  $(-1, -1)$ , B =  $(\frac{1}{2}, 2)$  and C =  $(4, \frac{1}{4})$ .



### Vectors as Matrices

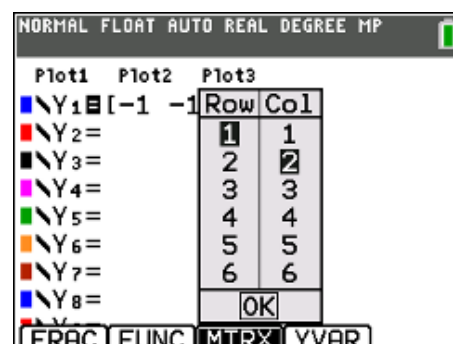
A vector in **component** form can be used to reference point A:

$$OA = \underline{a} = -\underline{i} - \underline{j}$$

One way to represent a vector on the calculator is to use a matrix. This vector can be stored in Matrix A; however, for this investigation it will be stored directly into  $Y_1$ .

Press  $\boxed{y=}$  to access the editor, followed by  $\boxed{\alpha}$  and  $\boxed{r3}$  for the Matrix insertion shortcut.

Use this menu to insert a  $1 \times 2$  matrix to represent the vector OA.



Use the above process to create vectors OB and OC in  $Y_2$  and  $Y_3$  respectively.

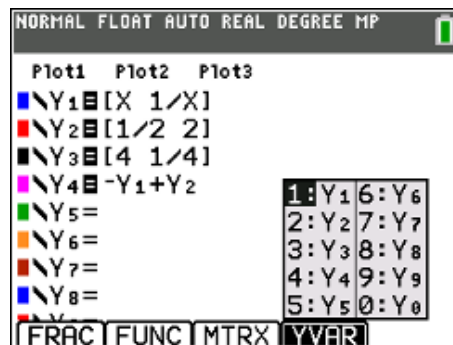
Vector AB is equal to:  $AO + OB$  or  $-OA + OB$

Vectors AB, BC and CA will be stored in  $Y_4$ ,  $Y_5$  and  $Y_6$  respectively.

Within the Y = editor, navigate to  $Y_4$  and press  $\boxed{\text{vars}}$ , arrow across to Y-VARS and select Function followed by  $Y_1$ , alternatively press  $\boxed{\text{alpha}} \boxed{[f4]}$  and select the corresponding function.

To calculate  $-OA + OB$ , enter the expression:  $-Y_1 + Y_2$

$Y_4$  now contains vector AB.



**Question 1**

On the calculator home screen use the  $\boxed{\text{vars}}$  key and function menu to recall  $Y_4$  and hence determine the matrix that represents vector AB. Explain your answer in terms of point A and B.

$Y_4 = [1.5, 3]$

Explanation: To get from point A (-1, -1) to point B (1/2, 2) move 1.5 units in the positive x direction and 3 units in the positive y direction.

**Question 2**

Determine expressions for vectors BC and CA in terms of OA ( $Y_1$ ), OB ( $Y_2$ ) and OC ( $Y_3$ ) as applicable. Store these expressions in  $Y_5$  and  $Y_6$  respectively. Write down the matrix and equivalent vector expressions for BC and CA.

$Y_5 = [3.5, -1.75]$      $BC = 3.5\hat{i} - 1.75\hat{j}$      $Y_6 = [-5, -1.25]$      $CA = -5\hat{i} - 1.25\hat{j}$

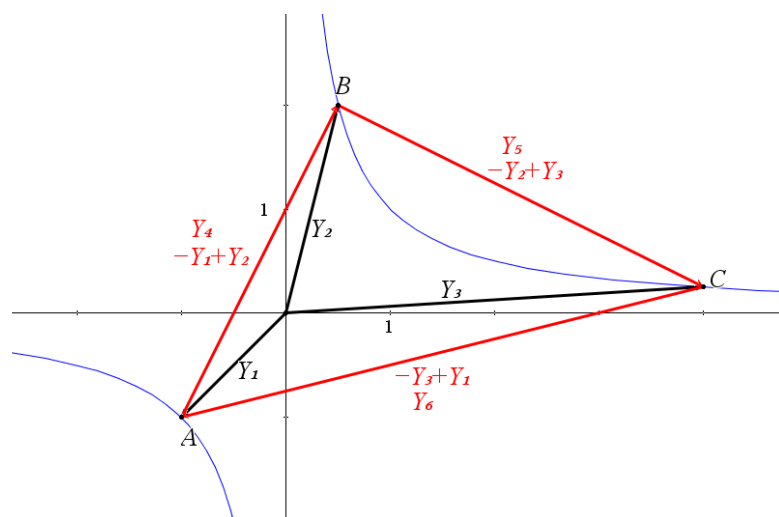
**Question 3**

On the calculator home screen calculate:  $Y_4 + Y_5 + Y_6$  and explain the result.

$Y_4 + Y_5 + Y_6 = 0$  As vectors:  $AB + BC + CA = AA = 0$ . The sum of the three vectors is for a closed figure, the triangle, so the 0 simply indicates that we are back to where we started from.

**Question 4**

Draw a diagram of the graph and all six vectors. (Label each vector as  $Y_1, Y_2 \dots Y_6$ )



### Magnitude of a Vector

The magnitude of vector is calculated using Pythagoras's theorem, squaring each component, adding these squared values together and then calculating the square-root:

The magnitude of a vector can be computed by:

$$|\underline{a}| = \sqrt{x^2 + y^2} \quad \text{where } \underline{a} = x\underline{i} + y\underline{j}$$

We can see how this works with matrices:

$$\underline{a} = [x \quad y]$$

To multiply matrix A by itself requires the matrix to be transposed so the rows and columns align:

$$(\underline{a})^T = \begin{bmatrix} x \\ y \end{bmatrix}$$

The product is therefore:

$$\underline{a} \times \underline{a}^T = [x \quad y] \begin{bmatrix} x \\ y \end{bmatrix} = [x^2 + y^2]$$

The determinant of this 1 x 1 matrix is simply  $x^2 + y^2$  which then allows the square-root of this value to be computed.

**Note:** The determinant and transpose commands are located in the matrix menu:  $\boxed{2\text{nd}}$  [Matrix] > MATH

```
NORMAL FLOAT AUTO REAL DEGREE MP
NAMES MATH EDIT
1:det(
2:T
3:dim(
4:Fill(
5:identity(
6:randM(
7:augment(
8:Matr>list(
9↓List>matr(
```

```
NORMAL FLOAT AUTO REAL DEGREE MP
√det(Y4*Y4T)
..... 3.535533906
```

### Mathematics

#### Tip!



In “vector world”, the magnitude of a vector can also be computed using the dot product:

$$|\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}}$$

### Question 5

Determine the magnitude of vectors AB, BC and CA and hence the perimeter of triangle ABC.

$$\frac{3\sqrt{5}}{2} \approx 3.354$$

$$\frac{7\sqrt{5}}{4} \approx 3.913$$

$$\frac{5\sqrt{17}}{4} \approx 5.154$$

$$\text{Perimeter: } \frac{5\sqrt{17}}{4} + \frac{13\sqrt{5}}{4} \approx 12.421$$

Suppose point A (-1, -1) is moved to a new location on the curve  $y = \frac{1}{x}$ , the perimeter of the triangle would obviously change.

Use the Y = editor to set the coordinates of point A as:  $\left(x, \frac{1}{x}\right)$

Using a variable in the matrix can only be achieved whilst the matrix is associated with a Y= equation. This approach allows the matrix to be used in a 'solve' command.

Suppose point A is moved such that the perimeter is now equal to 17. This means that the sum of the side lengths, vector magnitudes, will be equal to 17.

$$|AB| + |BC| + |CA| = 17$$

The numerical equation solver can be accessed by pressing:

$\text{math}$  > Option C (Numeric solver)

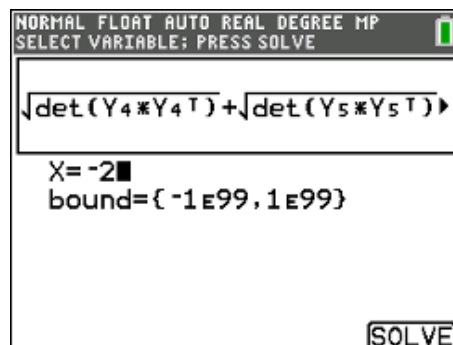
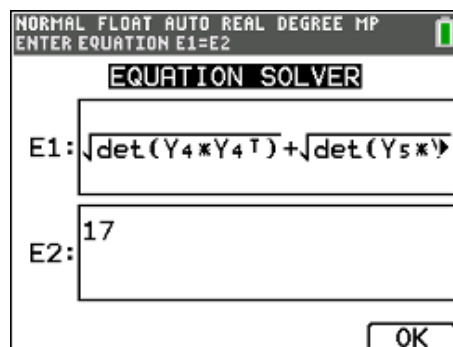
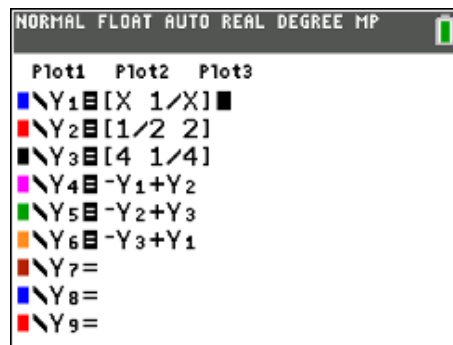
Enter expressions for the magnitude of each vector for E1 (Equation 1) and the perimeter for E2 (Equation 2) as shown.

In this menu the  $\text{GRAPH}$  key becomes a 'soft' key for the "OK" option.

Select "OK", edit the approximate starting value for X to direct the calculator where a solution may be located.

Select "SOLVE" ... the calculator will search for an answer, as indicated by the 'spinning wheel' adjacent to the Battery symbol.

This process may be hastened by reducing the default boundary conditions:  $\{-1 \times 10^{99}, 1 \times 10^{99}\}$



For large and repetitive calculations it can be more efficient to store an expression as a 'string'. Place quotation marks around the expression and use  $\text{sto}$  and  $\text{vars}$  > String to store.

The mathematical expression can then be 'recalled' by using the 'recall' operation (RCL) located above  $\text{sto}$  and selecting the corresponding String variable from the  $\text{vars}$  menu.

**Question 6**

What location(s) are possible for point A to generate a perimeter of 17?

Point A could be any of the following:

$$x_a = -4.0216 \text{ and } y_a = -0.24866 \quad \text{OR} \quad x_a = -0.21682 \text{ and } y_a = -4.6121$$

$$x_a = 0.13967 \text{ and } y_a = 7.1595 \quad \text{OR} \quad x_a = 8.6854 \text{ and } y_a = 0.11514$$

Note that two solutions exist in the 3<sup>rd</sup> quadrant and two in the 1<sup>st</sup> quadrant.

**Teacher Notes:**

The geometric interpretation of this result is worth considering. The length BC remains unchanged while distances AB and AC can change but only so that the perimeter remains constant. This means that  $AB + AC = \text{Constant}$ , this is a construction technique for producing an ellipse with  $F_1$  and  $F_2$  located at points B and C. Draw an ellipse over the hyperbola  $y=1/x$  and it will intersect at four places. It is worthwhile illustrating this concept to students using the TI-Nspire software.

It is crucial that students consider the domain over which to search for solutions and therefore edit the boundary conditions or by careful selection of the 'guess' (original value) for point A. Using the boundary conditions students can purposely exclude points that have already been found.

This particular question highlights the power of using Vectors on the calculator, the algebra to solve this problem is very complicated and beyond the scope of high school mathematics courses.

**Dot Product of a Vector**

The dot product of two vectors is computed as follows:

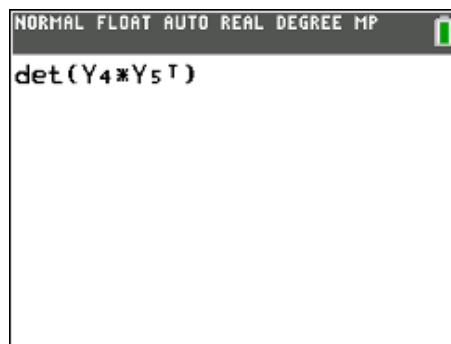
$$\underline{a} \cdot \underline{b} = x_a x_b + y_a y_b + z_a z_b \text{ where: } \underline{a} = x_a \underline{i} + y_a \underline{j} + z_a \underline{k} \text{ and } \underline{b} = x_b \underline{i} + y_b \underline{j} + z_b \underline{k}$$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos(\theta) \text{ where } \theta \text{ is the angle between the two vectors.}$$

The dot product of two vectors can be used to compute the angle between the vectors, in this section however we will use the fact that if the dot product is equal to zero then the vectors are at right angles. ( $\cos(90^\circ)=0$ )

It is easy to calculate the dot product of the vectors stored in  $Y_4$ ,  $Y_5$  and  $Y_6$ , the vectors that define the sides of the triangle.

$$\det(Y_4 * Y_5^T)$$

**Question 7**

Change the location of point A to: (-1, -1) then calculate the dot product of vectors AB and BC ( $Y_4$  and  $Y_5$ ). What does the result say about the angle between these two sides?

The dot product is equal to zero, this means that AB is perpendicular to BC.

**Question 8**

Using your answer to question 6 to help determine the area of the triangle ABC where A is located at (-1, -1).

$\text{Area} = \frac{1}{2}bh$  Students could compute this value using the magnitude of vectors  $Y_4$  and  $Y_5$ .

$$\text{Area} = \frac{1}{2} \times 195/8 = 6.5625$$

Suppose the vertices of the triangle are now:

$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = \left(c, \frac{1}{c}\right)$$

Point C is to be moved such that a right angled triangle is to be formed once again.

Edit  $Y_3$  to include 'c'. Note that  $c \neq 0$ .

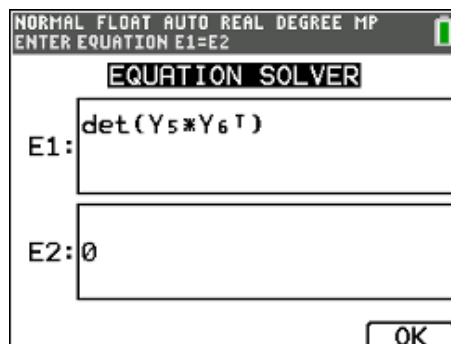
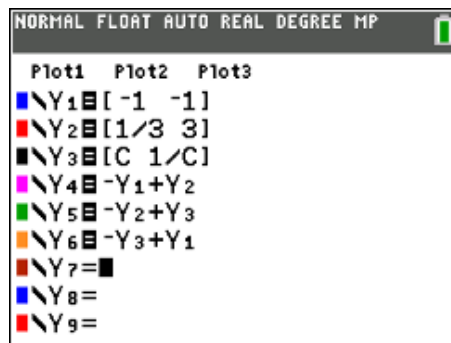
The calculator's numeric solver can now be used to locate point C.

We are looking for vectors BC and CA to be perpendicular, therefore

$$BC \cdot CA = 0 \quad \text{or} \quad \det(Y_5 * Y_6^T) = 0$$

This can be entered into the numeric solver: Math > Numeric Solver

E1 represents Expression 1, similarly E2 represents Expression 2.

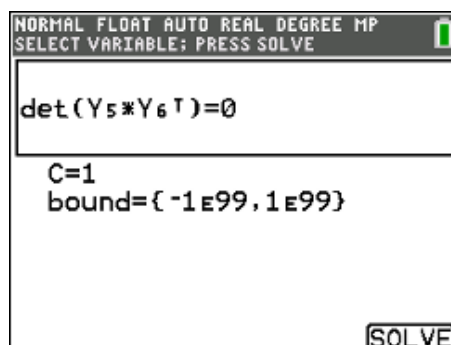


The equation to be solved is:

$$\text{Expression 1} = \text{Expression 2} \quad (\text{as shown})$$

Note that the value entered in C is the current value stored in C. If this current value is zero, then it will need to be changed before attempting to solve the problem since  $C \neq 0$ .

If the process is slow, the boundary conditions can be narrowed manually to expedite the computations or if specific values are sought.



### Question 9

What is the value of C for which ABC is once again a right angled triangle?

Note: This can be expressed as an exact value.

$$c = \sqrt{3}$$

### Question 10

Verify that the value of C forms a right angled triangle.

This can be done a number of ways, Pythagoras's theorem is one way.

$$|AB| \approx 4.2164, \quad |BC| \approx 2.7974 \quad \text{and} \quad |CA| \approx 3.1547 \quad |AB|^2 = |BC|^2 + |CA|^2$$

### Calculator

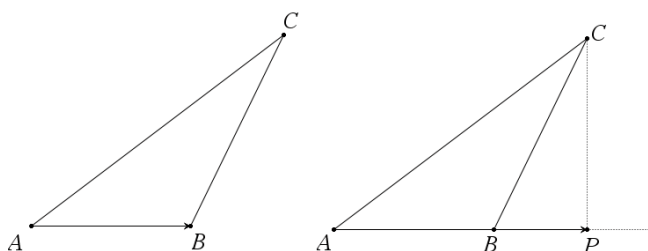
#### Tip!



After using the Numeric Solver command, the computed value is automatically stored in the corresponding variable. This means that the vectors BC ( $Y_5$ ) and CA ( $Y_6$ ) will now be equal to the vectors that form a right angled triangle.

### Vector multiplied by a Scalar

When a vector is multiplied by a scalar, the direction<sup>1</sup> of the vector remains the same but the length will change if the scalar quantity is not equal to 1. A simple way to explore this is through a geometric example.

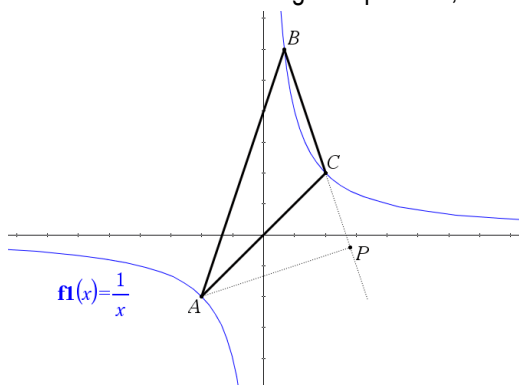


Consider the triangles shown here. When trying to calculate the height of  $\triangle ABC$  we can extend the base to point P such that  $\angle APC = 90^\circ$ .

Vector AB can be multiplied by a scalar  $n$  such that vector  $AP = nAB$ .

Vector AP has the same direction as AB but has a different length. **Note:** PC is the altitude of  $\triangle ABC$ .

For this section of the investigation points A, B and C are defined as:



$$A = (-1, -1) \quad B = \left(\frac{1}{3}, 3\right) \quad C = (1, 1)$$

Notice that BC has been extended such that  $BP = nBC$ .

P is located such that PA is perpendicular to BP.

This means that  $AP \cdot BP = 0$

**Note:** AP is the altitude of  $\triangle ABC$ .

#### Question 11

Write a vector equation for AP in terms of BC and AB.

$$AP = AB + nBC$$

#### Question 12

Determine the value of  $n$  such that  $AP \cdot BP = 0$

**Note:** The numerical solver on the calculator can be used to determine the value of  $n$ .

$$\text{Solution: } n = 1.6$$

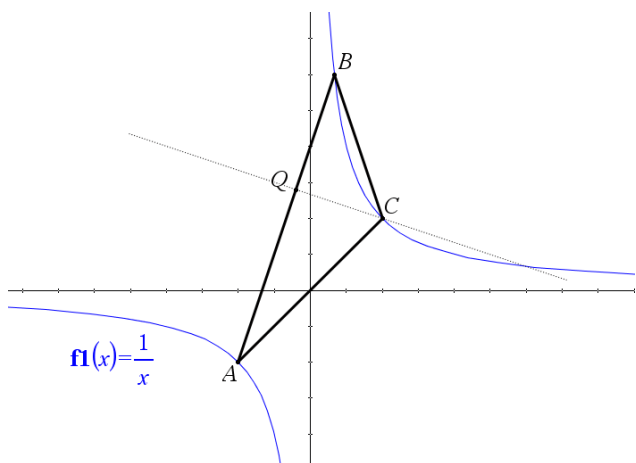
#### Question 13

Determine the vector OP and hence the coordinates of point P.

$$OP = OB + nBC$$

$$OP = [1.4, -0.2]$$

<sup>1</sup> Direction – The direction will reverse if the scalar is a negative quantity.



$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = (1, 1)$$

Point Q is such that AQ is perpendicular to CQ.

**Note:** CQ is another altitude of  $\triangle ABC$ .

#### Question 14

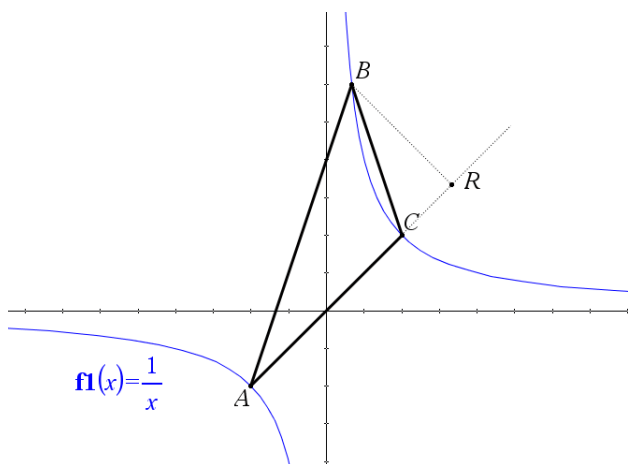
Write a vector equation for CQ in terms of CA and AB.

$$CQ = CA + mAB$$

#### Question 15

Determine the value of  $m$  such that  $AQ \cdot CQ = 0$  and hence the vector OQ.

$$M = 0.6 \text{ therefore } OQ = OA + mAB = [-0.2, 1.4]$$



$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = (1, 1)$$

Point R is such that AR is perpendicular to BR.

**Note:** BR is the third altitude of  $\triangle ABC$ .

#### Question 16

Write a vector equation for BR in terms of AB and CA.

$BR = -AB - sAC$  Note that students may write this expression as  $-AB + sAC$  and therefore obtain a different value for  $s$ .

#### Question 17

Determine the value of  $s$  such that  $sAC \cdot RB = 0$  and hence the vector OR.

$$S = 4/3. \text{ OR} = OA + sAC = OA - sCA = [5/3, 5/3]$$

#### Question 18

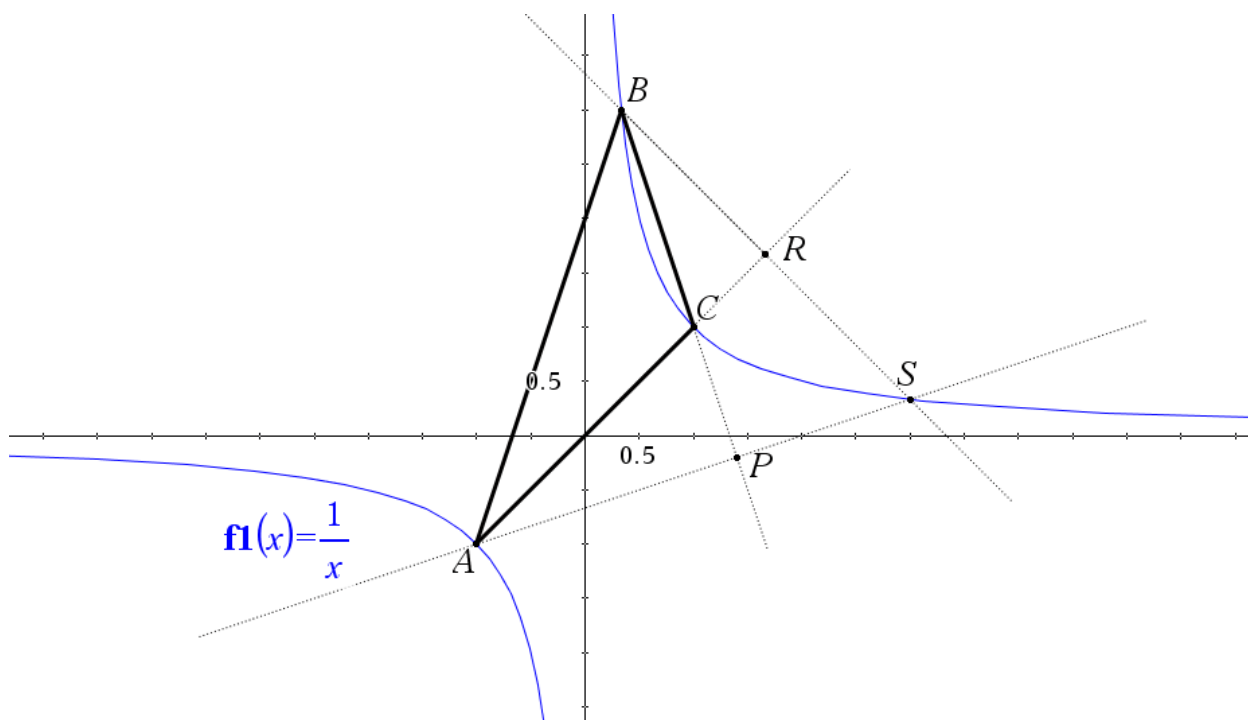
Consider the values of  $m$ ,  $n$ , and  $s$  established in the previous questions and their respective locations. By considering these values only, how is it possible to tell if the altitude (perpendicular) sits on the vector?

If the scalar quantity ' $n$ ' is such that  $|n| > 1$  then the point is beyond the vector, a projection in the direction of the original vector.



### Extension: Points of Intersection Investigation

It is possible to determine where each of the altitudes intersects one another. The diagram below shows the point of intersection between altitudes BR and AP at point S.



#### Important knowledge:

- $\vec{BS}$  is a scalar multiple of  $\vec{BR}$ .
- $\vec{AS}$  is a scalar multiple of  $\vec{AP}$ .
- $\vec{OS} = \vec{OB} + \vec{BS}$  and  $\vec{OS} = \vec{OA} + \vec{AS}$ .

The above information can be used to determine the location of  $\vec{OS}$ . Determine the location of S and explore the location of the point(s) of intersection for the altitude passing through C with those passing through A and B.

Vary the locations of points A, B and C and summarise the location of point S and the general vector defining OS.

#### Established already:

$$\vec{OR} = [5/3, 5/3] \text{ so } \vec{BR} = \vec{BO} + \vec{OR} = -\vec{OB} + \vec{OR} \text{ therefore } \vec{BR} = [4/3, -4/3]$$

$$\vec{OP} = [1.4, -0.2] \text{ so } \vec{AP} = \vec{AO} + \vec{OP} = -\vec{OA} + \vec{OP} \text{ therefore } \vec{AP} = [2.4, 0.8]$$

Vector OS can be written as:

$$\vec{OS} = \vec{OA} + p\vec{AP} \quad \text{or} \quad \vec{OS} = \vec{OB} + r\vec{BR}$$

These equations can be solved simultaneously (relatively simple... so appropriate to do by hand).

$$p = 5/3 \text{ and } r = 2$$

Vector OS is therefore:  $[3, 1/3]$  ... which places it on the function  $y = 1/x$ . This is true for the intersection of the altitudes of the triangle with vertices on the function  $y = 1/x$ . Therefore, regardless of where students place the points A, B and C, vector OS will always point to the function.

For students that are particularly keen ... this can be proved algebraically!